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Investigation of Sixth-Grade Students' Cognitive Processes in a Learning Trajectory Designed for Basic Geometric Constructions

Ömer Deniz [1], Tangül Kabael [2]

Abstract

In this study, it is aimed to design a learning trajectory that supports the conceptual infrastructure and dynamic ways of thinking in the process of basic geometric constructions of sixth-grade students and to examine the cognitive development of the students in this process. In this qualitative study, which is a design based research, data were collected through individual and group worksheets, video recordings of courses, field notes, homework sheets and clinical interviews with focus participants. The focus participants were determined by criterion sampling through an open-ended test developed for the concepts of point, line, line segment and ray, which were considered as prerequisites for the basic geometric constructions. The study was conducted in three phases which are preparation and design, teaching experiment and retrospective analysis. During the preparation and design phase, the initial hypothetical learning trajectory has been designed with the literature review performed to allow comprehensive analysis in epistemological and didactic aspects. Formative evaluations were made by micro analyzes during the teaching experiment carried out for the implementation and evaluation of the learning trajectory. Finally, comparative analysis was conducted on students' thinking processes and actions between the assumed learning trajectory and the actual learning trajectory. Consequently, the revised learning trajectory and students' developmental progress are presented in an interpretive framework. In the most general sense, supporting the cognitive actions such as realizing a construction in different ways and directions, taking into account the changing and unchanging aspects of the geometric structures during the construction process, interpreting the variability of the compass opening, revealing all of the possible points as a circle or part of the required points as an arc, making changes in the route of the steps and interpreting or defending these changes are important in building dynamic geometric constructions. It was noted that the students who could follow the algorithmic steps only operationally without making defense with mathematical justifications were able to put forward

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[1] Tuna Imam Hatip Secondary School, Turkey, omerdenizmat@gmail.com
[2] Anadolu University, Faculty of Education, Dept. of Mathematics and Science Education, Turkey, tuygur@anadolu.edu.tr

more static thinking processes. It was seen that learning environments limited only by analysis and construction phases might be insufficient in realizing constructions with strengthened conceptual infrastructure, and therefore, the importance of designing learning environments that provide opportunity for proof and discussion phases that support dynamic processes of thinking was revealed. In addition, it has been concluded that a learning trajectory which provides opportunity to take into consideration the characteristic features of the geometric structures in the constructions and to make connections with other structures or constructions makes important contributions in strengthening of the conceptual infrastructure.

Introduction

Geometry skill is a basic mathematical skill that requires making sense of shapes in relation to measurement and problem solving skills, establishing relationships between their properties and parts (Sherard, 1981). Supporting this skill is directly related to geometric thinking (Muhassanah, 2014, as cited in Nur & Nurvitasari, 2017). Geometric thinking is constantly evolving as follows: recognizing shapes perceptually, realizing the characteristic features of shapes and establishing relations between shapes based on these features, moreover, revealing patterns between shape families and between different axiomatic systems by reasoning on shape properties. In supporting the cumulative development of geometric thinking processes of a learner, it is important to create environments that allow them to make connections between geometric shapes, features, and systems while overcoming a problem situation. Geometric construction problems require not only visualizing shapes but also analyzing geometric structures and establishing relationships between these structures. In this way, it allows the application of important and critical properties of geometric structures in different problem situations and creates a great driving force that contributes to the development of thought (Stupel & Ben-Chaim, 2013). Geometric constructions contribute to the clarification and strengthening of many geometric relations in the minds (Sanders, 1998). The process of geometric construction, which means the building of geometric structures with the help of certain geometric tools, requires individuals to make judgments such as explaining, proving and expanding their actions on the results they observe or achieve (Köse, Tanışlı, Erdoğan, & Ada, 2012). The critical point in this process is not to be able to draw a shape randomly, but to produce precise algorithmic steps on how to draw it only with the help of compass and straightedge (Smart, 1998). However, in this process, not only the construction of a geometric structure, but also how this construction is realized is accepted by the mathematicians as a problem situation (Erduran & Yeşildere, 2010). Therefore, it is important to carry out actions that allow the making sense of the steps taken such as explanation, justification and proof, rather than memorizing the steps leading to a definitive solution in a geometric construction process. Besides, when necessary, reflecting previous steps and constructions in different new construction problems by making dynamic transitions between the steps, contributes significantly to supporting geometric thinking. If students are expected to make their own explanations and proofs of the steps they take during the process of the construction, the learning environments to be provided to them should be carefully designed (Kondratieva, 2013). Especially for complex and higher level geometric constructions that are expected to be carried out at high school level, internalizing Euclid's basic constructions is essential (Kondratieva, 2011; Napitupulu, 2001). According to mathematics curriculum for primary and middle schools published by the Ministry of National Education (Milli Eğitim Bakanlığı, MEB) in 2018, students encounter basic geometric constructions for the first time in the middle school years. For this reason, studies on the process of realizing the basic constructions have gained importance since these years. In this study, it is aimed to design learning trajectories that include possible cognitive actions that shed light on the developmental progress of students in a learning environment based on the realization of basic geometric constructions at the sixth grade level.

Geometric Construction and Related Literature

It is seen that geometric constructions are positioned in the center of geometry in the "Elements" book of Euclid, written around 300 BC, which forms the basis of plane geometry (Martin, 2012). It is known that in Ancient Greece, where strict rules exist, geometric constructions are allowed to be realized only by using straightedge and compass and only constructions made with the help of these two tools are accepted (Albrecht, 1952; Hogben, 2004). Smart (1998) defines geometric constructions as problem solving strategies with the help of a set of rules including a series of basic and complex steps. According to Smart (1998), it is aimed to construct a geometric structure on the basis of its characteristic features or relations with other structures by using compass and straightedge rather than just drawing in geometric construction problems. In the literature, it is emphasized that geometric construction activities contribute positively to the development of students' geometric thinking levels (e.g. Cheung, 2011; Güven, 2006; Napitupulu, 2001; Uygun, 2016). Geometric constructions realized by compasses and straightedge contribute directly to the development of cognitive skills by supporting understanding of the exact features of geometric shapes beyond the development of psycho-motor skills (Cheung, 2011; Napitupulu, 2001). From this point of view, it is not considered sufficient to perform geometric constructions only operationally. Such a process is expected that mathematical justifications of the steps can be made and conceptual infrastructure is strengthened. In this way, it is thought that the contribution of geometric constructions to the individual can be increased to higher levels. On the other hand, students generally have difficulty applying their geometric knowledge to problem situations (Kondratieva, 2011). Köse et al. (2012) emphasize that geometric concepts, shapes, features and proofs are memorized without making sense, and this causes difficulties in learning geometry, and that geometric construction activities can provide an important opportunity to overcome this negativity. Euclid's basic geometric constructions have a critical importance since they are seen as the basic stepping stones in solving and proving complex geometrical construction problems (Kondratieva, 2011; Napitupulu, 2001). According to Smart (1998, p. 166), these basic constructions can be listed as follows: (i) constructing a congruent segment, (ii) finding the middle point of the segment, (iii) constructing a line perpendicular to a given line at a point on the line, (iv) constructing a line perpendicular to a given line from a given point not on the line, (v) constructing a triangle when given three sides, (vi) constructing angle bisector, (vii) constructing a congruent angle, (viii) constructing the triangle when given the two sides and the angle between them, (ix) constructing a line parallel to a given line.

In the literature, it is seen that some of the studies that focus on the realization of these basic geometric constructions with compasses and straightedge were carried out with mathematics teachers or teacher candidates, and the other with high school and middle school students. Erduran and Yeşildere (2010), who observed three mathematics teachers in their teaching environments, concluded that they were not aware of geometric constructions or they thought of them as activities that only serve to visualize or they did not need to realize these constructions at all during teaching. In another study in which four mathematics teachers participated, it was observed that geometric constructions in classroom activities could only be handled superficially (Öçal & Şimşek, 2017). Also Öçal and Şimşek (2017) showed that unwanted actions such as using a ruler instead of a straightedge, skipping some steps while following sequential steps, which may lead to a decrease in the anticipated contributions by the realization of the construction, can be revealed by teachers. In the literature, it is seen that only one of the other studies on geometric construction activities performed using compass and straightedge was conducted with high school students (Cheung, 2011). The rest is observed to be conducted with prospective mathematics teachers. In a study conducted with 18 high school students on average of 15 years old, Cheung (2011) concluded that geometric construction activities carried out with the help of compass and straightedge support students' critical ways of thinking and directly contribute to their verification and proofing skills. Tapan and Arslan (2009) aimed to determine how teacher candidates use mathematical features and visual elements in the process of realizing geometric constructions. As a result, it was seen that most of them used visual elements and made experimental justifications while performing geometric constructions (Tapan & Arslan, 2009). Karakuş (2014) examined the views of 63 elementary mathematics teacher candidates who took a course for geometric formation in the geometry lesson at the university, on geometric construction activities. As a result, they revealed that they think these activities can help to understand geometry topics. In this study, it was observed that teacher candidates had difficulties in deciding which steps to take in which order while performing geometric constructions with the help of compasses and straightedge. Uygun (2016) has designed a hypothetical learning trajectory that involves performing geometric constructions with the help of compasses and straightedge in order to improve the subject matter knowledge of mathematics teacher candidates about the triangle concept. As a result, it was concluded that the learning trajectory applied in the argumentation-supported learning environment improved their subject matter knowledge and contributed to their leap in geometric thinking levels (Uygun, 2016). In the study of Gür and Demir (2017), in which 72 high school mathematics teacher candidates participated, it was concluded that performing geometric constructions using compasses and straightedge allowed them to increase their geometric thinking levels and also revealed that their attitudes towards mathematics changed positively in this process. In another study in which high school mathematics teacher candidates participated, Stupel and Ben-Chaim (2013) studied constructions that require the application of a trapezoid theorem using only straightedge. Stupel and Ben-Chaim (2013) demonstrated that the applied trapezoid theorem can be used as a tool to construct a line perpendicular using only straightedge. Kuzle (2013) conducted teaching experiments on teacher candidates from two different cultures to realize geometric constructions with different tools such as compasses and straightedge, protractors, and Dynamic Geometry Software (DGS) in two separate groups. In this study, the researcher examined the teacher candidates' opinions about both the change in the process of understanding as a learner and the benefits they will provide to students within the framework of geometric thinking from the teaching they will provide in this direction as a teacher in the future (Kuzle, 2013) As a result, all participants resorted to the use of compasses and straightedge during the realization of geometric constructions and defended their importance by drawing attention to the role of these tools in establishing connections between geometric relationships, as well as their contribution to mathematical reasoning in the study of Kuzle (2013). Another important result obtained from the study of Kuzle (2013) was that the participants found DGS useful in geometric constructions in terms of providing visualization, accuracy and fastness. Moreover, they highlighted the importance of incorporating DGS in learning environments due to their critical role in the validation process.

When the literature is examined, it has been seen that geometric constructions have gained a different dimension with the emergence of DGS and a tendency towards studies focusing on the realization of geometric constructions in the DGS environment (e.g. Çiftçi & Tatar, 2014; Kondratieva, 2013; Köse et al., 2012; Laborde, 2005; Lukáč, 2010; Pratt & Ainley, 1997; Stylianides & Stylianides, 2005; Ulusoy, 2019; Yıldız, 2016). There are also studies examining the use of DGS with compasses and straightedge. Among these studies, Çiftçi and Tatar (2014) made a comparison by examining the effect of DGS and the use of compass-straightedge on the success of teacher candidates in the realization of geometric construction activities. As a result, Çiftçi and Tatar (2014) concluded that both of them affect success positively and it was concluded that there was no significant difference between the two in terms of affecting success. Köse et al. (2012) examined the effect of a teaching process supported by TI-Nspire CAS calculators including Cabri Geometry II software on their ability to realize the desired geometric constructions with a geometry drawing set consisting of a ruler, protractor, compass and square. Köse et al. (2012) observed that the participants, who were observed to perform invalid constructions by adhering to a single feature of the desired construction in the pre-test, were able to make theoretical constructions by taking into account more than one geometric feature in the post-test applied after technology-supported teaching. Ulusoy (2019) focuses on the parallel line construction processes of middle school mathematics teacher candidates. In this study, firstly, constructions were realized with compasses and straightedge on paper, and then they were verified with dynamic geometry software (GeoGebra) and given the opportunity to discuss. As a result, the teacher candidates showed that they were aware of the contribution of DGS supported learning environments in realizing alternative constructions and verifying them in the study of Ulusoy (2019). However, Ulusoy (2019) concluded that they realized that they had to provide well-grounded reasons to defend the construction. There are also studies in the literature in which software programs that can be referenced in realizing or proving geometric constructions are designed (e.g. Djoric & Janicic, 2004; Gulwani, Korthikanti, & Tiwari, 2011; Janičić, 2006; Kellison, Bickford, & Constable, 2019).

In the literature, there are several studies conducted with middle school and high school students on geometric construction activities using compasses and straightedge. In a study conducted with 18 high school students aged 15 years on average, Cheung (2011) concluded that geometric construction activities performed with the help of compasses and straightedge supported students' critical thinking ways and directly contributed to their development of verification and proofing skills. Tosun (2019) revealed that during the drawing of the angle bisector, students had the most difficulty in making the construction through compasses and straightedge in the study examining the mathematical thinking processes of ninth grade students towards the concept of angle bisector. Ulusoy (2014) applied a test to 498 middle school students for their concept images and misconceptions about perpendicularity and parallelism. As a result, Ulusoy (2014) showed that the images that middle school students create for steepness in their minds are affected by the visual images they have in relation to the prototype shapes. In another study by Ulusoy (2016), in which 83 students from sixth and seventh grade participated, it was concluded that the sample spaces limited by the effect of their visual images caused the students to not give exact and correct answers while making decisions about steepness and parallelism. In Ulusoy's (2014, 2016) studies, it was seen that the images of middle school students for the concept of perpendicularity included the idea that two vertical lines should average each other, that the lengths of the lines are limited, that the vertical lines should be equal in length, and that only a pair consisting of only one horizontal and one vertical line can form perpendicularity. In another study focusing on the concepts of perpendicularity and parallelism, Paksu and Bayram (2019) conducted a case study on the processes of sixth grade students to determine and construct perpendicularity and parallelism., It was observed that the students were more successful for horizontal and vertical lines while constructing the perpendicular and parallel lines in the study of Paksu and Bayram (2019).In addition, it was concluded that the appearance of the line pairs was taken as basis for the students to decide whether the lines are perpendicular or parallel (Paksu & Bayram, 2019). In another study in which middle school students were participants, Chikwere and Ayama (2016) conducted an experimental study with 60 middle school students. In the pretest for the realization of geometric constructions, it was seen that the students had little knowledge of geometric constructions and 87% of them failed. Then the participants were divided into two different groups and one group was taught based on the abstract method and the other on the practical method. As a result, it was concluded that the knowledge and understanding of the students participating in the teaching process based on the practical method developed better than the other group in the study of Chikwere and Ayama (2016). Lim (1997), who claimed that questioning the reason of the steps taken in the process of geometric construction and making connections between constructions improve high-level thinking skills, presented an example activity scheme in this direction. This activity scheme indicates a teaching process that is carried out on the route of determining the properties of an isosceles triangle, determining the properties of deltoid and rhombus, and building angle bisector and perpendicular bisector by using the properties of rhombus or isosceles triangle. It has been suggested by Lim (1997) that this scheme can allow students to make meaningful constructions and thus contribute to higher-order thinking skills by supporting relational understanding.

It is seen in the literature that geometric construction activities contribute to the individual in many different ways, such as supporting relational thinking, accelerating the transition to higher-level geometric thinking, providing opportunities to demonstrate mathematical thinking skills such as verification and proof. In order to make this contribution more efficient, it is emphasized that it is not enough to realize the geometric constructions only operationally, while attention is drawn to the necessity of learning environments that require questioning processes such as being aware of what the steps taken mean, which features of which geometric structures are used and how relationships are made between them. Lim (1997) argues that students in schools cannot understand why geometric constructions are performed, and emphasizes that geometric constructions are taught as an operative process consisting of a series of complex steps and that no justification is given as to why and how the steps taken in this process. In the literature, it is seen that middle school and high school students have difficulty in realizing geometric constructions and cannot defend the actions they put forward in this process. Moreover, it is seen in the literature that teachers and teacher candidates have similar difficulties. It is noteworthy that they generally emphasize only the operational aspects of the constructions and are aware of their contributions such as enabling visualization and encouraging tools. Although it is not expected to reveal formal thinking processes such as proof in middle school years when basic geometric constructions are expected to be realized, students should be encouraged to make justifications in the teaching process and they should be provided with meaningful bases (Kunimune, Fujita, & Jones, 2010; Lim, 1997). In this way, an important contribution will be made to go beyond the fact that the geometric structure takes place in the mind of the individual only visually, and that it is associated with other structures and is ready to be reflected in a different problem situation by passing through different thinking filters. Geometric constructions require the reflection of many different geometric constructions that have been constructed before (Kondratieva, 2011; Lim, 1997). From this point of view, it is not enough to realize the basic geometric constructions, which are frequently reflected in many constructions, only operationally from early years. Strengthening and supporting the conceptual infrastructure is important for the developmental progress that is expected to continue regarding geometric constructions in the following years. With the increasing use of technology and DGS in education, important opportunities have arisen for teachers and students to dynamically explore geometric relationships and to create more complex geometric constructions (Stupel & Ben-Chaim, 2013). However, many of the DGS have buttons that directly provide basic constructions such as drawing line perpendicular, constructing line parallel, finding midpoint, constructing congruent segment. Considering that the individual performs these basic constructions with only these buttons from an early age, it is clear that she/he will have difficulty in realizing them in the physical environment and defending them in the virtual environment. Therefore, there is a need for an instructional design that supports the conceptual infrastructure of geometric constructions that are neglected in learning environments or that are only allowed to be realized operationally. This study has an original value in terms of revealing a learning trajectory design for teaching basic geometric constructions. In addition, in the literature, it is seen that more studies are needed to reveal the cognitive processes of middle school students while they realize geometric constructions. In this study, which is aimed to reveal an instructional design for basic geometric constructions in the sixth-grade, the cognitive development of the students in the designed learning trajectories is investigated, thus contributes to filling an important gap in the literature. Since the teaching designed in the research is aimed to support the conceptual infrastructure related to geometric constructions and to investigate the processes of realizing the basic geometric constructions of Euclid by sixth-grade students in this learning environment, the tools in geometric constructions were chosen as a compass and straightedge only. The five geometric constructions considered as learning goals in the research are seen in Table 1. Based on these learning goals, the following question was explored:

How is the cognitive development of sixth grade students in a learning environment guided by a learning trajectory designed to support the realization of basic geometric constructions?

Table 1. Learning Objectives Targeted in The Study

Learning Goal 1	Construct a congruent line segment with the help of compasses and straightedge.
Learning Goal 2	Construct a triangle given three sides with the help of compasses and straightedge.
Learning Goal 3	Find the midpoint of a line segment with the help of compasses and straightedge.
Learning Goal 4	Construct a line perpendicular to a given line from a given point not on the line with the help of compasses and straightedge.
Learning Goal 5	Construct a line perpendicular to a given line at a point on the line with the help of compasses and straightedge.

Theoretical Framework

The theoretical framework of this research has been determined as the Hypothetical Learning Trajectory (HLT). Because it was predicted that the learning environment, which was designed according to the goals and the epistemological characteristics of the concepts with which these goals are related, would need to be revised according to the cognitive development of the students. HLT was first proposed by Simon (1995) as a model for how the constructivist approach in mathematics education can be reflected in the learning process. Accordingly, HLT involves the learning goals, the design of instructional activities that may enable these goals, making assumptions about the development of the learning process including the implementation of these activities. The design of instructional activities that play a critical role in achieving the goals and the hypothetical learning process, which includes the implementation of these activities, are intertwined (Simon & Tzur, 2004). The hypothetical learning process requires predicting how students' thinking and understanding will evolve in the context of learning activities (Simon, 1995). Both epistemological and didactic analysis supported by the literature and thought experiments play a critical role in predicting the possible learning process. According to Simon (1995, 2014), a learning environment in which the interaction between teacher and students is at high level is required in teaching experiments that provide a basis for the functionality of the activities in the possible learning process. In addition, it is important to make ongoing observations in order to reveal the predicted paths in the conjectured possible learning trajectory throughout the teaching experiments (Simon, 2014). The comparison of the consistency of student actions with the conjectured learning process by revealing the changes in students' understanding and thinking, learning difficulties and errors experienced in the process, and other variables affecting learning, helps to distinguish between the conjectured and actual learning trajectories in achieving the targeted learning goals (Mavrotheris & Paparistodemou, 2015). According to Clements and Sarama (2004), the main purpose in this theoretical framework is to "reveal possible trajectories that include a series of teaching activities that allow students to develop developmentally towards the targeted learning goals and possible cognitive processes and actions that may occur in the implementation of these activities" (p. 83). These trajectories are presented to the field as renewable, iterable and improvable by revising them according to the cognitive actions of the students during the teaching experiment.

Method

This qualitative study, which aims to examine cognitive development, is a design-based research since it involves designing a learning environment (Bakker & Van Eerde, 2015). Design-based research, which is defined as designing and developing an intervention to provide a solution to a complex educational problem in the most general sense, also aims to increase the knowledge about the characteristics of these interventions and their design and development processes (Plomp, 2007). Although the design-based research is carried out in a limited set of constructed setup, they do not aim to examine only the processes that support new learning occurrences in this specific setup. At the same time, it aims to put forward an iterable local instructional theory that is open to continuous improvement for possible processes that frame the selected aspects for envisaged learning and support learning (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). According to Gravemeijer (1999), a local instructional theory is a framework that includes a set of educational activities that reveal and support the possible mathematical development of students for a concept aimed to be structured, and at the same time, serves as a reference source for teachers. The local instructional theory requires demonstrating learning goals, teaching activities, and the use of tools (e.g. compasses and straightedge) that support the learning process in the envisaged learning path (Gravemeijer, 2004). In this study, it is aimed to design a learning trajectory seen as a local instructional theory that reveals the cognitive actions of students towards the process of realizing basic geometric constructions at the sixth-grade level.

Participants

In this study, the teaching experiments based on the proposed hypothetical learning trajectory were conducted with sixth-grade students of a rural public school. Students who received consent forms from their parents voluntarily participated in the research. The number of these participating students is 12 in total. One of the researchers is a mathematics teacher who has 12 years of teaching experience and is about to finish his doctorate in mathematics education. The other researcher is an academician specializing in mathematics education. One-to-one clinical interviews were conducted after the first three teaching experiments and at the end of the teaching experiments in order to collect in-depth data on the cognitive development of the students in the process of realizing the basic geometric constructions. The focus participants with whom these interviews were conducted were selected by criterion sampling, one of the purposeful sampling methods (Yıldırım & Şimşek, 2018). Before the research, an open-ended test was prepared for the concepts of point, line, line segment, and ray, which are considered as prerequisite knowledge for sixth-grade students to realize basic geometric constructions. This test contains questions such as "What does a line segment mean in mathematics (in geometry)? How would you define it? Can you draw a line segment in the space given below?" Students were divided into four groups in line with their performance in this test (Table 2). Their performance in the pre-test was accepted as a criterion. From each of these groups, a random student representing the group was determined as the focus participant. The nicknames of these participants used in this study are Emre in the first group and Ceylan, Salim, and Ilkan, respectively, towards the fourth group.

Table 2. Groups Formed in Line with Different Performances as a Result of the Analysis of the Pre-Test Developed for Basic Geometric Figures

Group	Performance
1st group	S/he is aware that a point occupies a place on the plane and comments that it is the building block of geometric shapes. S/he also shows awareness of its size. (e.g. *Everything is a point. For example, the beginning of a line is a point. The thickness of a point has no length*) S/he draws a line and supports her/his drawing with explanations about the properties of the line. (e.g. *the line is a geometric shape that goes from both ends to infinity*) S/he draws a line segment and supports the drawing with explanations of the line segment properties. (e.g. *The line segment is a geometric shape limited at both ends*) S/he draws a ray and supports the drawing with explanations on the properties of the ray and can provide examples that can be considered correct from daily life. (e.g. *the ray is geometric shape that has a starting point and goes from the other end to infinity. For example, the sun or laser beam*)
2nd group	S/he explains the point through examples that are geometric objects. However, s/he cannot comment on its size. S/he also does not have any awareness that the point can be considered a building block for geometric shapes (e.g. *the point is used to indicate the vertices of the triangle*) S/he makes explanations about the properties of the line and the line segment, but has confusion about which shape is which. (e.g. *S/he draws a line segment instead of a line and explains it "goes from both ends to infinity")* S/he draws a ray and supports its drawing with explanations on the properties of the ray. (e.g. the ray is a geometric shape with a starting point that goes from the other end to infinity)
3rd group	Explain point by emphasizing a mathematical function on non-geometric objects. (e.g. The point is a symbol used in multiplication) S/he does not give any explanation about the properties of a line or line segment, and s/he has confusion about which shape is which. However, s/he makes explanations based on the existence of different geometric shapes. (e.g. When asked to draw a line, s/he draws line segment. When asked to draw a line segment, s/he draws a triangle and defends with the explanation "something that unites a triangle") S/he draws a ray as a line segment and, due to the knowledge s/he has gained from her/his daily life, it is seen that s/he can be partially successful in explaining the ray according to its properties (e.g. "The ray goes forever but stops if an obstacle comes in its way")
4th group	Explains point over non-mathematical objects. (e.g. the point is the sign put at the end of the sentence) S/he draws the line but offers false explanations for it. (e.g. When asked to draw a line draws a parallelogram and explains "lines go parallel") S/he draws a line segment but cannot support it with any explanation. S/he draws a ray but cannot support it with any explanation.

Data Collection Process, Tools and Analysis

Teaching experiments conducted in this research lasted 12 hours in total. Of the learning objectives presented in Table 1, four hours are allocated for only the fourth, while two hours are reserved for the others. Teaching experiments were carried out by the researcher, who was also the mathematics teacher of the class. For the first three teaching experiments, constructing the congruent segment, constructing a triangle when given three sides and finding the midpoint of a line segment were aimed respectively. Afterwards, one-to-one clinical interviews were conducted in order to enable the cognitive actions of the focus participants in the process of realizing these constructions in more detail. These interviews were conducted by the researcher, who was also a teacher. First of all, in the interviews, it was requested that the formations targeted in the teaching experiments be realized. Then, by presenting the differing situations, the process of realizing the construction was focused, and they were asked to justify their actions throughout the process. Also, inquiries were made about whether they could reveal alternative ways, whether they were aware of the relationships between basic constructions and geometric structures. During the interviews, the focus participants were given only paper, pencil, compasses, and straightedge, and were asked to think aloud while performing the requested constructions. Thus, it is aimed to collect richer data about their thinking processes. For example, during the interview, a student thinking out loud while constructing a perpendicular line made explanations such as *"Let's draw an isosceles triangle here, then find the midpoint of this (base)"*. These explanations are seen as an indication that s/he was aware that s/he built an isosceles triangle in the construction of the perpendicular line and reflects the construction of midpoint finding. Probe questions were posed such as *"why do you open your compass so much? How does it benefit you from using the compass?, Why do you do it like this?, How do you know what you did is right?, Can you do it differently?, What makes this point or points critical? Can you explain it? Can you find different points? Is there any other shape that you use as a tool while performing this construction? Can you show it? Can you explain?"* and enough time was given to answer them. In this way, in-depth data was obtained for the questions that were required to examine the cognitive development such as whether the focus participants memorized the actions they put forward in the teaching experiment only as a set of sequential steps by heart, whether they were aware of which step they took, whether they could explain what it meant by offering alternative ways for the step, whether he can make experimental justifications on the basis of only visual elements or whether he can defend it with justification pointing to the mathematical based conceptual infrastructure, whether he is aware that he may have reflected other constructions while performing a construction (e.g. reflecting the construction of congruent segments while constructing a triangle when given three sides), or whether it could reveal the existence of its relationship with other geometric structures while constructing the geometric structure (e.g. the emergence of an isosceles or equilateral triangle according to the compass opening during the finding of the midpoint of a line segment). The reason why the teaching experiment towards the fourth learning goal, which was carried out after the first clinical interviews, was planned as four hours is the allocation of extra hours for the process of building the perpendicular line to the base in an isosceles triangle, which forms the basis of the construction of the perpendicular line to a given line at a point on the line or not on the line. This teaching experiment involves an expanding process of constructing an isosceles triangle, then constructing the perpendicular line segment from its vertex to its base, and then reflecting this process for constructing the perpendicular line to a given line from a given point not on the line. Finally, the last two-hour teaching experiment aimed at reflecting a similar process for constructing the perpendicular line to a given line at a point on the line was conducted. Then, final clinical interviews were conducted with the focus participants. The research process is shown in Figure 1 in the most general framework.

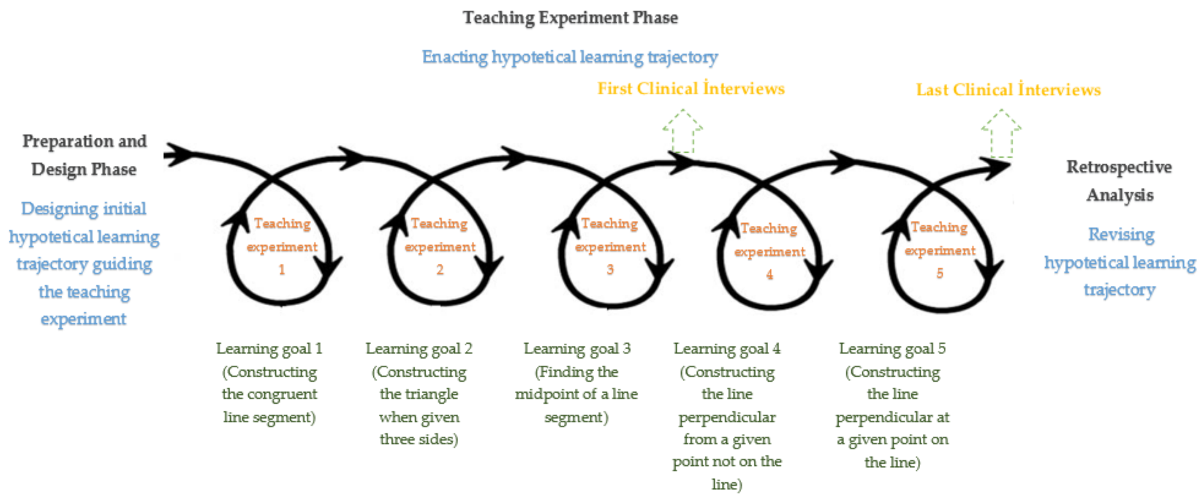

Figure 1. The Research Process

Throughout the teaching experiment, individual and group worksheets in which the students noted their actions such as their strategies, starting and destination points, exit routes, and the reasons for the steps they took were used for data collection. The tasks presented to the students in the lessons were usually conveyed to them verbally by the teacher, and initially, they were asked to perform these tasks on blank and white worksheets. In this way, they were allowed to reflect their basic geometry knowledge. For example, the task was transferred to them on the following route: *"First, let's construct a line on the plane. Now let's set a point that does not belong to this line. Let's name the point. Now, let's build the perpendicular line to this line that I want from you, but you have determined a point that does not belong to the line, or you know, the perpendicular line must pass through that point. Can you do that?"*. In this process, they are given the opportunity to use basic geometry knowledge such as constructing a line, determining a point, naming the point, realizing that the line consists of infinite points, and being aware of the existence of points that do not belong to the line. Later on, such tasks, which were initially presented step by step, were transferred directly to them: *"Build a perpendicular line to another line that is not horizontal or vertical in the plane. Make sure that the point where the perpendicular line will pass is not on the line. Explain the steps you have taken in this process with their reasons"*. Each lesson conducted in teaching experiments was recorded with a video camera, and field notes were kept by the researcher, who is the teacher, throughout the process. Considering that the teaching experiment was recording with a single camera to see the whole class, it was predicted that there would be a loss in the data collected by video recordings. The explanations recorded on individual worksheets and these field notes kept contributed significantly to minimizing data loss. For example, in the video recordings of the lesson for the construction of the congruent line segment, the process of realizing the construction of the student individual could not be followed. The explanation in the worksheets that some students construct the congruent line segment by marking only on the straightedge provided data about the cognitive process they performed for this construction. However, some students did not have any explanation in this direction on their worksheets. However, in the field notes taken by the teacher, it was seen that some of these students verbally stated that they used the compass opening, but they determined only two points instead of showing all possible points. Such explanations and actions predicted to be critical were noted by the teacher-researcher in the teaching process. In this way, it was aimed to obtain in-depth data about the cognitive processes of those students. Also, the homework papers given to the students after the teaching cycles served as a tool for data collection. In the homework papers, they were asked to perform construction processes by giving questions similar to the tasks in the course. Students were encouraged in terms of making justified explanations and revealing alternative ways. For example, the following tasks were included in the homework paper given after the course on finding midpoint: *"Find the midpoint of the given line segment (by giving the horizontal, vertical and diagonal line segment)"*, *"construct an equilateral triangle and find the midpoints of all sides on the same triangle"*, *"The line segments given below are* *the sides of a triangle. Accordingly, find the midpoints of the sides after constructing the triangle with these sides"* A total of eight clinical interviews, two times with each of the focus participants, were used as a data collection tool to allow in-depth analysis of cognitive developments. At the end of the teaching experiment, the video recordings of the course and the interviews were transcribed, and then the data obtained on the thinking processes and actions of the students were analyzed using the constant comparative analysis method (Bakker, 2004; Glaser & Strauss, 1967). Firstly, the obtained data are divided into sections considering their chronological order. The initial learning trajectory and learning goals presented in Table 4 guided the determination of these sections. Sentences, expressions, explanations, actions deemed valuable in terms of the research question were coded in individual and group worksheets, course video recordings, and interview transcripts for each section. After these codes were categorized and recurring themes were reached. Then, the consistency of the codes under these themes with the default actions in the initial learning trajectory was tested. Bakker (2004) explains testing as "seeking confirmations and counter-examples" (p. 45). The consistency of the coding made by two different researchers separately for each section was examined, and 93% consistency was determined. The analysis of the data obtained from the interviews was decisive in terms of providing in-depth information in making a judgment about the inconsistent parts. For example, it was observed that a participant transferred the first line segment based on a single point in the construction of a triangle given three sides. However, during the interview, s/he showed that s/he was aware of other possible points when s/he saw it necessary or when asked directly. This action played a critical role in the judgment that s/he was in a dynamic thinking process. Field notes and homework were analyzed qualitatively using content analysis method (Yıldırım & Şimşek, 2018). The results obtained from these data collection tools also contributed significantly to making judgments about inconsistent parts. For example, in the construction of a triangle given three sides, a field note about a focus participant who determines all possible points as a circle includes the explanation, *"In fact, there is no need to draw the entire circle, teacher, it is enough to draw the intersecting parts, but I still drew it".* This data served as an important indicator in terms of showing that the focus participant was able to construct the arc by deciding on some of the possible points. From this point of view, without the data obtained from field notes, it would be concluded that s/he could only construct a circle while determining possible points about this student. It would be judged that s/he was not yet aware that s/he could reveal enough of the possible points with the construction of the arc.

According to Pelczer, Singer, and Voica (2014), static thinking is the automatization of actions or thoughts and the application of only standard algorithms and operations in situations encountered as a result of this. Dynamic thinking, on the other hand, is the ability to change actions and thoughts in a way to adapt to different situations, to be able to look from a wider perspective to see the negligible aspects of a situation, and thus to offer different interpretations and suggestions, to be able to act more freely in mental situations such as seeing shapes in different positions and interpreting their changing unchanging aspects (Pelczer, Singer, & Voica, 2014). Some of the performances of the participants that are interpreted as indicators of the static thinking process in this study are as follows: memorizing a set of steps without being able to make a mathematical justification, being able to realize the construction only for certain situations (e.g. finding the midpoint of only horizontal line segments), not being able to perform alternative constructions or steps (e.g. seeing only one of the intersection points that provides the construction while constructing an equilateral triangle), being unable to realize the construction or losing control when faced with a differentiating situation (e.g. when constructing a perpendicular line from a point not on the line, in cases where the line needs to be extended, not being able to perform any action or an attempt to make the construction with similar steps without being extended). Some of the more flexible thinking actions that are interpreted as indicators of dynamic thinking process are as follows: being able to perform the construction and mathematically justify actions for differentiating situations, the ability to demonstrate and defend different actions for the same construction (e.g. showing and defending that there may be different compass openings when the midpoint is found), to be able to see the changing and unchanging aspects of the construction in differentiating examples, to reveal and justify the possible different situations that lead to the realizing of the construction. In line with the results obtained after the teaching experiments, the revisions deemed necessary for the initial hypothetical learning trajectory were made. Besides, the accuracy of the results achieved was strengthened with worksheets, homework, and field notes. For example, it was seen that a participant used only one of the intersection points of the circles in the construction of midpoint in the teaching process. However, when both the individual worksheet and the field notes were examined, it was seen that s/he was aware of the other point, s/he used the liner to pass through those two points and could defend these actions. Besides, it was noteworthy that s/he determined the midpoint based on two intersections in her/his homework. The participant, who found the midpoint based on the two points where the circles intersect during the interview, showed that s/he could defend the critical property of these points and even reveal different points with similar characteristics. In this way, the reliability of the study has been strengthened by performing data triangulation (Yıldırım & Şimşek, 2018).

Design-based studies consist of three basic phases: preparation and design, teaching experiment, and retrospective analysis (Bakker & Van Eerde, 2015). In this section, information about the procedures carried out throughout the research is presented under the headings based on these three basic phases.

Preparation and Design Phase

In this study, the literature review was deepened to allow comprehensive analysis in terms of epistemological and didactic aspects in order to design the initial hypothetical learning trajectory for the realization of the basic geometric constructions of Euclid at the sixth-grade level. Besides, how these basic constructions are handled in the mathematics curriculum and textbooks were examined. Moreover, the researches about what the difficulties students have in this subject might be, what they can learn, and how they can improve were examined. In this way, in the process of achieving the targeted learning goals, the students' possible cognitive and physical actions were shed light and prepared for possible educational intervention. In the most general sense, this process can be explained as follows: for example, middle school textbooks and middle school mathematics curriculum (MEB, 2018) published by the Ministry of Education for the 2018-2019 academic year were examined for the construction of the midpoint of the line segment. It has been observed that the realization of the construction with the help of compass and straightedge is included. However, it was noteworthy that only the show-and-make method was used. It is also seen in the literature that in these and similar constructions, teachers either use this method or do not make such constructions at all, and even they are not aware of what the steps they take mean (e.g. Erduran & Yeşildere, 2010; Karakuş, 2014; Lim, 1997; Öçal & Şimşek, 2017; Ulusoy, 2019). Therefore, an epistemological examination has been undertaken in order to reveal the conceptual infrastructure of this construction in order to predict what tasks and how the teacher can support the students in making sense of these steps. It has been provided to find answers to questions such as the critical feature of the midpoint for that line segment, what the compass opening means and how much it should be, the critical property of the intersection points of the circles formed in the same compass opening, and what the line formed by critical points (perpendicular bisector) means for this construction. These critical actions, which were predicted to allow the meaning of the steps taken in the realization of the formation, directly contributed to the design of the initial hypothetical learning trajectory. In line with the literature review, which was expanded for the learning and teaching process of the targeted construction, the hypothetical learning trajectory was finalized by referring to the views of a teacher who has 11 years of teaching experience, who is also continuing her doctorate education in mathematics education, and an academician who has studied in geometry education. Tasks, which have a critical role in achieving the targeted learning goals, were designed by taking into account the possible student learning and thinking with the help of the researchers' experiences and thought experiments. The steps of Smart (1998) in the realization of geometric constructions guided the design and route of these tasks (Table 3).

Table 3. Possible Steps in The Realization of Geometric Constructions (Smart, 1998 p.168)

1. Analysis	In this step, the solver first assumes that s/he has realized the construction, and then analyzes the completed picture of the result to provide the needed links between the facts given in the original problem and the unknown elements in the figure.
2. Construction	In this step, the shape is constructed with the compass and straightedge and the construction traces are shown.
3. Proving	It must be proved whether the resulting shape is really the desired shape.
4. Discussion	The number of possible solutions and the conditions for any possible solution are discussed in this step.

The initial hypothetical learning trajectory, which includes the main tasks designed and the possible cognitive and physical actions expected by students in line with these tasks, is presented in Table 4.

Table 4. Initial Hypothetical Learning Trajectory for The Learning Process of Basic Geometric Constructions in the Sixth Grade

Initial Hypothetical Learning Trajectory
Learning Goal 1: Constructs the congruent line segment with the help of compass and straightedge.
Task 1: Draw a line segment congruent to the given line segment. Build a line segment and transfer it anywhere on the plane.
Conjectured Learning Process ♠ Realizing that the lengths of the line segments to be created or existing will be equal. ♠ Using the opening of the compass to determine the length of the line segment to be transferred. ♠ Determining two points that are as far away from each other with the help of a compass and constructing the line segment between these points.
Task 2: Transfer the given line segment to point A on the plane in such a way that point A is one of the endpoints. (Note: A is a point not on the line).
Conjectured Learning Process ♠ Creating a compass opening in the length of the line segment to be transferred and determining the other point, which is far enough from point A to provide the congruent line segment, by using this compass opening. ♠ Constructing the line segment between point A and the other point determined as far away as the length of the line segment to be transferred.
Task 3: Show the possible line segments that can be constructed congruent to the given line segment, with point A (another point not on the line) being one of the endpoints. Show the possible points for the other endpoint providing the congruent line segment provided that point A remains constant.
Conjectured Learning Process ♠ Creating a compass opening in the length of the line segment to be transferred and determining the possible points as a circle for the other endpoint by using this compass opening, provided that point A is one of the endpoints.
Main Critical Actions Expected in the Learning Process ♣ S/he realizes that in the transfer of the line segment, one of the extreme points can be taken randomly and the other can be taken as a point as far as the line segment from that point. ♣ S/he accepts that it is forbidden to mark on the straightedge and determines a point at a certain distance using a compass ♣ S/he can reveal possible points that are at a specified length from a point by the construction of a circle.

Table 4. Continued

Initial Hypothetical Learning Trajectory

Learning Goal 2: Constructs the triangle when given three sides with the help of a compass and straightedge.

Task 4: The three line segments given are the sides of a triangle. Construct this triangle.

Conjectured Larning Process

- ♠ Realizing that for the triangle construction to be created, the given line segments need to be transferred in such a way that their endpoints are common.
- ♠ Transferring any line segment on the plane to the place where the triangle construction will take place.
- ♠ Transferring the other line segments to the endpoints of the first transferred line segment.
- ♠ Determining possible points as far as the other line segments separately from the endpoints of the first transferred line segment with the help of a compass.
- ♠ Considering one of the intersection points of the resulting circles as critical because it provides the transfer of both line segments, determining this point as the third corner, and consequently constructing the desired triangle.

Task 5: (In relation to Task 4) You found the third corner of the triangle. Are there any other points that could be the third corner? If yes, show it.

Conjectured Learning Process

- ♠ Being aware that the intersection point of the circles containing possible points constructed to transfer other line segments to the endpoints of the first transferred line segment is symmetrical based on the first transferred segment.

Main Critical Actions Expected in the Learning Process

- ♣ S/he reflects the construction of the transference line segment by determining the possible points.
- ♣ S/he realizes that finding the desired critical point by the trial-and-error method is both challenging and doubtful.
- ♣ S/he sees that the intersections of the circles are critical, but s/he also realizes that not every intersection point is critical for achieving the desired construction.
- ♣ S/he knows that s/he can transfer one of the line segments randomly, and makes sense that she must transfer the others to the endpoints of the randomly assigned line segment.

Learning Goal 3: Finds the midpoint of a line segment with the help of a compass and straightedge.

Task 6: Find the midpoint of the given line segment.

Conjectured Learning Process

- ♠ Realizing that the midpoint of the line segment is equidistant from the endpoints.
- ♠ Determining a certain distance and revealing the possible points that are as far away as the specified distance from the endpoints with the help of a compass (building circles with a specified radius, with the endpoints being the center).
- ♠ So realizing that the intersection points of the circles are critical since they are equidistant from the extremes.

Task 7: (In relation to Task 6) Can you use the compass with a different opening? How many different points can you find equidistant from the extremes? (Points forming the perpendicular bisector are being questioned)

Conjectured Learning Process

- ♠ Finding other critical points (intersection points of circles) that are equidistant from the endpoints by changing the opening of the compass, seeing linearity of them, and constructing the line consisting of these intersection points (construction of the perpendicular bisector).
- ♠ To be able to determine the intersection point of the line segment and the perpendicular bisector as the midpoint of the line segment.
- ♠ To be able to interpret that the opening of the compass can be infinite, provided that it is longer than half of the line segment.

Table 4. Continued

Initial Hypothetical Learning Trajectory
Main Critical Actions Expected in the Learning Process ♣ S/he notices the set of points that are equidistant from the endpoints (the perpendicular bisector) and sees that the midpoint is only one of these points. ♣ S/he makes sense that determining the tangent point of the circles as the critical point or the sought point is considered unacceptable because the tangent point cannot be revealed exactly. ♣ S/he realizes that critical points can be obtained by the intersection of circles as well as by the intersection of lines. ♣ S/he realizes that the common property of being equidistant from the endpoints remains unchanged despite the variation of the compass opening.
Learning Goal 4: Constructs the line perpendicular to a given line from a given point not on the line with the help of the compass and straightedge.
Task 8: Draw an isosceles triangle.
Conjectured Learning Process ♠ Determining a point that will be the vertex of an isosceles triangle and revealing possible points equidistant from that point by constructing a circle with a compass. ♠ Constructing an isosceles triangle, realizing that any two of the points on the circle can be taken as base vertices.
Task 9: Determine the base of an isosceles triangle on a line yourself. Then create an isosceles triangle of this base. **Task 10:** Identify a point not on the line. Construct an isosceles triangle where this point is the vertex and the base is a line segment on the given line.
Conjectured Learning Process ♠ Determining the possible points that are equidistant from a point not on the line with the help of a compass. ♠ Realizing that two of these points lie on the line, building an isosceles triangle where they are the base vertices.
Task 11: Construct the line segment that accepts the vertex of the isosceles triangle and the midpoint of its base as the endpoints.
Conjectured Learning Process ♠ Determinig the midpoint of the base of an isosceles triangle by reflecting the midpoint construction. ♠ With the help of a straightedge, building the line segment that accepts the midpoint of the base and the vertex as the endpoints.
Task 12: In an isosceles triangle, show the height descending from the vertex to the base in accordance with the information that the perpendicular (height) descending from the vertex to the base divides the base into two equal parts.
Conjectured Learning Process. ♠ Realizing that the line segment where the vertex and the midpoint of the base are the extreme points is the height (perpendicular line). ♠ Being aware that the midpoint of the base must be found when it is requested to build the perpendicular line from the vertex to the base.
Task 13: A line and a point not on the line are given. From this point to the line, build the perpendicular line.

Table 4. Continued

Initial Hypothetical Learning Trajectory
Conjectured Learning Process ♠ Putting forward thinking processes to relate the process of building a perpendicular line segment (height) from the vertex to base in an isosceles triangle with building a line perpendicular to a given line from a point not on the line. ♠ Identifying possible points that are equidistance to a point (vertex) not on the line by constructing a circle or arc with the help of a compass and being aware of the fact that the line segment between these two points of them on the line is the base ♠ Determining the midpoint of the base by reflecting the process of finding the midpoint and constructing the line segment between the vertex and the midpoint of the base as the height. ♠ Recognizing that the constructed height is a line segment and being able to construct the line that includes that segment if desired. ♠ While initially constructing the equilateral sides of the isosceles triangle, in later times, realizing the construction of the perpendicular line without revealing these sides. Referring to it as a mathematical justification for defending the steps of this construction, even if the isosceles triangle is not clearly revealed.
Task 14: (in relation to Task 13) What do you think about how much compass opening should be when constructing a perpendicular line from a point not on the line? Show it.
Conjectured Learning Process ♠ Being aware of the fact that for the circle or arc whose center is the point not on the line, provided that the compass opening is wide enough to intersect the line at two points, there may be infinite different lengths. ♠ In the construction of finding the midpoint of the base, being aware of the fact that the compass opening can be infinitely different in length, provided that it is more than half of the line segment that is the base.
Task 15: Build a non-horizontal line. Set a point not on this line. Can you construct another line that passes through the point you specified and is perpendicular to that line? **Task 16:** Construct the perpendicular lines passing through designated points not on the given lines (non-horizontal lines).
Conjectured Learning Process ♠ Being able to construct the perpendicular lines to a non-horizontal line from points taken in different places. ♠ Being aware of more flexible actions such as demonstrating that can construct different isosceles triangles during the construction process, interpret the variability of compass opening, being able to extend the line when necessary. Moreover, being able to defend these actions with mathematical reasons.
Main Critical Actions Expected in the Learning Process ♣ S/he defends any step or element in the construction with the following theorem: In an isosceles triangle, the base of the perpendicular line descending from the vertex to the base divides into two equal parts. ♣ S/he reflects the construction of finding the midpoint of the line segment and realizes that the vertex remains on the perpendicular bisector. ♣ S/he realizes that she can obtain critical points from the intersection of two circles or two lines and that she can obtain them by intersecting the circle and the line.
Learning Goal 4: Constructs the line perpendicular to a given line from a given point on the line with the help of the compass and straightedge.
Task 17: A line and a point on this line are given. Construct the perpendicular line to this line through the given point.

Table 4. Continued

Initial Hypothetical Learning Trajectory
Conjected Learning Process ♠ Putting forward thinking processes to associate the process of building the perpendicular line segment (height) from vertex to base in an isosceles triangle with building the line perpendicular to a given line from a point on the line. ♠ Determining possible points as far as any compass aperture from the given point on the line with the help of a circle or arc. Being aware that two of these possible points on the line are critical since they are the endpoints of the base of the isosceles triangle. ♠ Determining a point not on the line that is equidistant to the endpoints of the line segment formed on the line and being aware that this point is the vertex of an isosceles triangle. ♠ Determine the possible points as a circle or arc, with the endpoints of the line segment on the line being the center, as far as the random compass opening. Being aware that one of the intersections of circles or arcs can be the vertex of an isosceles triangle. ♠ Building the line segment between the point determined outside the line (vertex) and the point on the line (the midpoint of the base) as the height with the help of a straightedge. ♠ While initially constructing the equilateral sides of the isosceles triangle, in later times, realizing the construction of the perpendicular line without revealing these sides. Referring to it as a mathematical justification for defending the steps of this construction, even if the isosceles triangle is not clearly revealed.
Task 18: (In relation to Task 17) What do you think about the compass opening when constructing the line perpendicular that passes through a point on the line? Show it.
Conjectured Learning Process ♠ Being aware that the compass opening for the circle or arc to be constructed provided that the point given on the line is the center can be infinitely different lengths without any limits ♠ Being aware that the compass opening may be infinitely different lengths for circles or arcs where the endpoints of the line segment formed on the line (base of an isosceles triangle) are the center, provided that the compass span is more than half of the line segment.
Task 19: Build a non-horizontal line and mark a point on it. Can you construct another line perpendicular to that line passing through the point you specified? **Task 20:** Build the perpendicular lines to the given lines (non-horizontal lines) passing through the specified points.
Conjectured Learning Process ♠ Being able to construct the perpendicular lines on a non-horizontal line from points taken in different places. ♠ Being aware of more flexible actions such as demonstrating that can construct different isosceles triangles during the construction process, interpret the variability of compass opening, being able to extend the line when necessary. Moreover, being able to defend these actions with mathematical reasons.
Main Critical Actions Expected in the Learning Process ♣ S/he defends any step or element in the construction with the following theorem: In an isosceles triangle, the base of the perpendicular line descending from the vertex to the base divides into two equal parts. ♣ S/he reflects the construction of finding the midpoint of the line segment by interpreting it in the direction of revealing the line segment that accepts a given point on the line as the midpoint. ♣ S/he can defend that any of the intersection points that appear in the formation of midpoint finding can be taken as a vertex by being aware that the vertex is on the perpendicular bisector.

Teaching Experiment

After revealing the initial hypothetical learning trajectory, the implementation and evaluation of this trajectory in the learning environment is accepted as the teaching experiment phase. The ongoing preliminary analysis carried out throughout the teaching experiment allows both the revision of the predicted trajectory towards achieving learning goals and the enrichment of possible interventions to be prepared for unexpected situations. For this purpose, the data obtained after every two hour periods were examined by the researchers. Formative evaluations were made on issues such as the functionality and revision of the conjectured learning process, the adequacy of possible interventions to overcome the difficulties encountered, and redefining the boundaries of guiding and questioning questions. The holistic analysis of all data obtained is carried out after the teaching experiment.

In this study, an environment based on cooperative learning where heterogeneous groups are involved in the learning process in which the teaching experiments are carried out was taken as the basis. Only the compass and straightedge tools and white paper were placed on the desk of each student. It was explained that they can use some of these papers as individual worksheets and some as group worksheets. In order to realize each assigned task, first of all, the opportunity was given to hold discussions within the group. Meanwhile, the teacher went through the groups, both following their actions and asking questions that prompted them to question or lead them to exploration. For example, the students' thinking processes were supported by inquiring and guiding questions such as *"what do you think does he mean when he calls the congruent line segment? Which property of the line segments must be the same to be congruent? What do you think of the aspects or directions of line segments?"* Then, in the intergroup whole-class discussions, one student from each group conveyed the actions of their groups, their results, their starting points, their strategies, and their reasons to the class. In each whole-class discussion period, attention was paid to the speech of different students representing their group. Besides, not only the students representing the group but also other students participated in the whole class discussions. Results obtained as a result of both in-group and inter-group whole class discussions were noted on individual and group worksheets, and these papers were collected by the teacher at the end of each lesson. In addition, homework assignments containing similar tasks were given after each lesson, and these homework papers followed by the teacher were collected. The teaching process is based on the following phases: *analysis, construction, proving, and discussion*. For example, in the process of constructing another line segment that is equivalent to a line segment, first thinking about what the demand for the "congruent line segment" requires and after discovering the criterion of equal length, drawing line segments of equal length by trial-and-error method or by eye decision indicates the *analysis* step. It is thought that constructing a congruent line segment by marking on the straightedge, which is seen as a prohibited action in the realizing of geometric formations by means of compass and straightedge, can be interpreted as a transition from the analysis step to the construction step. Determining the possible points (circle) that are as far as the line segment expected to be transferred from a point taken on the plane with the help of a compass, and combining one of these points with the center point to construct the line segment indicates the *construction* step. After the construction was realized, the process of questioning whether the two line segments were equal was started. The arguments such as demonstrating that the opening of the compass is the same for both, along with the justification that the line segments that seem possible to be revealed should be of equal length are the performances that are expected to be revealed in the *proving* step. Finally, in the question of whether it is possible to construct different congruent segments with the first point taken randomly on the plane (center point), showing and defending that any of the possible points built as a circle can be taken as the other endpoint of the congruent segment indicates the discussion step.

Retrospective Analysis

In the light of the data obtained from the teaching experiment, the goals, assumptions, and activities put forward at the beginning may be inconsistent. In this case, the proposed learning trajectory should be revised and refined on the basis of these inconsistencies and should be presented ready to be used and developed for further studies (Bakker & Van Eerde, 2015). Retrospective analysis requires working on all data sets obtained to contribute to developing a local instructional theory and an interpretative framework (Plomp, 2007). In the retrospective analysis phase, an interpretative framework that provides meaning to factors such as the confusion and clutter of classroom events is considered important (Gravemeijer & Cobb, 2006). In this study, in which the developmental progress in students' thinking processes was evaluated and presented in an interpretive framework, decisions were made regarding the revision of the learning trajectory in line with the results obtained. For example, in the initial learning trajectory, for the process of constructing the congruent line segment, it was envisaged that the students would use the compass opening to identify two points with a distance equal to this opening and construct the congruent line segment based on these points. In the teaching experiment, this prediction was realized, but it was seen that actions can be put forward by constructing the congruent lines only parallel to each other and with their endpoints one under the other. Therefore, the learning trajectory has been revised in this direction.

Results

In this section, the results will be presented under the heading of five basic geometric constructions to be constructed and parallel with the route in the learning trajectory. Considering that the constructions are structured in relation to each other, this route is considered important in terms of monitoring the developmental progress. Under each heading, firstly, quotations obtained from all participants during the teaching experiments were presented and the researcher was coded as a "teacher" while transferring the in-class discussions. Then, excerpts from clinical interviews are presented, which are related to the cognitive processes of the focus participants. While presenting the excerpts from clinical interviews, the researcher was coded as the "interviewer". Finally, under each heading, the hypothetical learning trajectory for that construction has been revised, if necessary. The revised parts of it are presented.

The Construction of Transferring the Congruent Line Segment

Faced with the task of constructing another line segment that is congruent to a given line segment, first of all, it has been observed that the students have made inquiries such as *"is it the same?", "Let's draw another one from this?"* to make sense of the concept of *"being congruent"* The teacher emphasized that in this process, which is seen as the analysis stage, the task also means transferring the line segment to another place on the plane. He asked them to discuss within the group what they understood from the expression "congruent" and received the answer "to be exactly the same". from each group. Then, the process of constructing the congruent line segment was started using a compass and straightedge. It was seen that most of the students construct a congruent line segment only with the help of a straightedge, parallel to the first line segment as their endpoints of them one under the other with the eye-decision method. Figure 2 shows the following explanation of a student who made the construction in this direction: *"I dropped two sticks from the first line segment I drew. Then I drew a straight line on its path with the help of a straightedge*". There have been also some students who constructed a congruent segment similarly by marking on the straightedge without using a compass. It has been observed that some of them have set out to realize more dynamic constructions by revealing congruent line segments in different ways and directions. In Figure 3, where quotations from the individual worksheets of the students who realized the construction in this direction are presented, the explanations of *"I made it by drawing a line on the straightedge"* and *"I put a mark on the straightedge and transferred it like that"* are shown. It is thought that this process, which is important for the making sense and implementation of the equal length condition required for the construction of the congruent line segment, is not appropriate to be seen as a construction phase due to the eye-decision or prohibited actions. It is correct to consider it as a transition from the analysis phase to the construction phase.

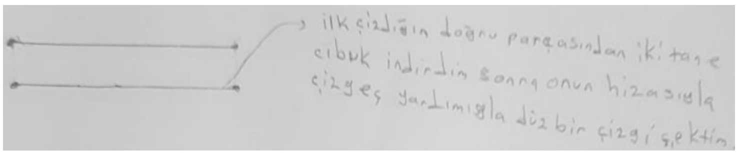

Figure 2. Construction of the congruent line segment with the eye-decision method as their endpoints one under the other.

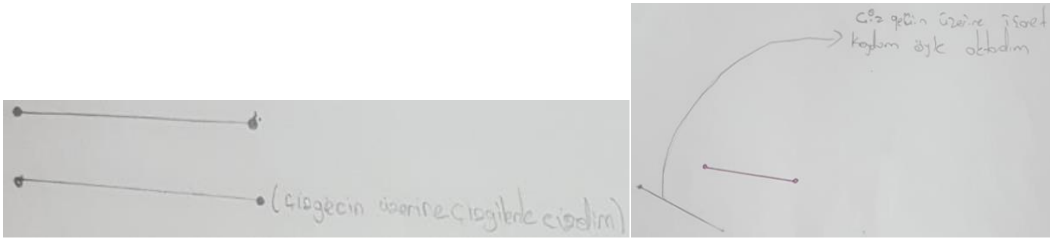

Figure 3. Constructing the congruent line segment with the endpoints one under the other by marking on the straightedge (on the left) and the construction of the congruent line segment in different ways and directions (on the right)

After drawing attention to the action of marking on the straightedge, which is forbidden in the constructions performed with the help of the compass and straightedge, a student said, *"I've already done it with compasses. I opened it to the length of the line segment and set the points with it"*. With this explanation, this action was deemed to be meaningful and acceptable by other students. It has been observed that some students were able to construct a congruent line segment with the help of a compass in different ways and directions (Figure 4). However, at this time, instead of determining the possible points, the students were able to construct the congruent line segment by only determining two points with a distance as much as the compass opening between them. It was observed that the students started to use the compass as a measuring tool with the question, *"why do you think these line segments are congruent?"*. It is thought that this inquiry process can be considered as the proving phase. Figure 4 shows the following explanation of a student: *"First, I measured the line segment I drew before with the help of a compass. So I opened my compass as much as the line segment. I put the endpoint and drew it with the help of a straightedge "*

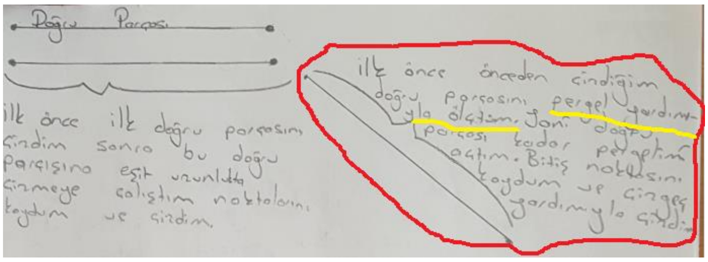

Figure 4. Construction of the congruent line segment in different ways and directions using a compass and the beginning to be seen as a measuring tool.

Although it points to dynamic thinking in which congruent line segments can be built in different ways and directions, the fact that the construction is not realized by determining the possible points suggests that the fully dynamic thought has not yet emerged. In order for the students to leap in this direction, they were directed to task 2 (transfer the given line segment to point A on the plane in such a way that point A is one of the endpoints) and task 3 (Show the possible line segments that can be constructed congruent to the given line segment, with point A being one of the endpoints. Show the possible points for the other endpoint providing the congruent line segment provided that point A remains constant.), which will require the construction and discussion phases, respectively. With these tasks, it was observed that the students were able to show that the possible points are infinite by constructing a circle with the help of a compass and then argue that an infinite number of congruent segments can be constructed (Figure 5). Also, in this process, which indicates the proving phase, students were asked questions such as: *"Is there any other line segment equivalent to this?, where are the other congruent segments?, what is the common feature of these points on the circle?, how many congruent segments can be drawn?, why do you think that?, can you show me?"*. In this way, they were encouraged to make a justified defense by revealing some of the congruent line segments. Figure 5 shows a student's argument that "I have determined eight points, but there can be infinite line segments because there are infinite points in the circle".

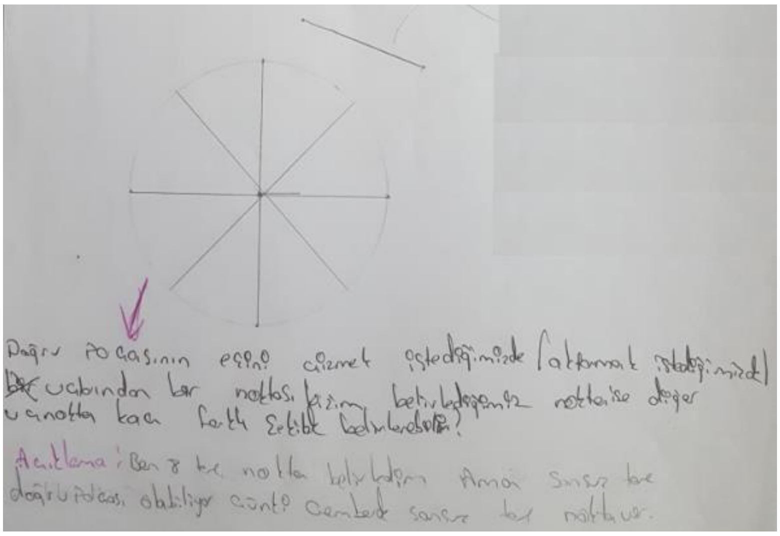

Figure 5. Realizing and defending the construction of the congruent line segment by determining the possible points that provide the congruence using a compass.

In one-on-one interviews with the focus participants, Emre has shown that he is aware that all possible points form a circle for the other endpoint by accepting any point on the plane as a fixed endpoint for the congruent line segment, and that even the arc construction, which is a part of the possible points, is sufficient. It is predicted that the circle construction, which includes all of the possible points, may evolve into an arc construction, which appears to be a sufficient part of the possible points. From this point of view, it is important for the developmental progress of this participant to start the arc construction at this moment. This participant, who can defend the congruence with the expressions of *"copy-paste and being the same"*, has not yet been able to defend why he built an arc at this moment with a justification that points to the dynamic thinking process such as determining possible points. Below, in the interview held with Emre, quotations regarding the explanations he made during the questioning of the construction of the congruent line segment are presented.

Interviewer: What have you done right now?

Emre: I did the same of that here. It's like a copy-paste.

Interviewer: How many congruent line segments can you draw?

Emre: Infinite. But I have to draw a circle (talking about completing the arc he draws as a circle).

Interviewer: So, why do you think you can draw infinite congruent line segments?

Emre: Because the circle forms infinite points. Since we draw a line segment between the center point and the point forming the circle, there are infinite number of line segments from the center point to here.

At the beginning of the interviews, Ceylan and Salim were observed to construct the congruent line segment with the help of the compass but by determining only one point without determining the possible points. However, it was seen that they could easily defend that an infinite number of congruent line segments can be constructed by questioning how many congruent segments can be built and how many of the other endpoints of the congruent line segment on the condition that one endpoint on the plane is fixed. Meanwhile, it was noteworthy that they could put forward the existence of infinite points on the circle as a mathematical basis. As a result, it was seen that these participants, beyond realizing the construction of the congruent line segment, could exist by putting forward mathematical reasons with the thinking processes that became dynamic in the proving and discussion phases. Below, Ceylan's explanations about the possibility of constructing infinite congruent line segments by showing the existence of infinite points on the circle as a basis are given.

Interviewer: How many of these line segments can you get?

Ceylan: Infinite. There is an infinite number of congruent line segments in this circle.

Interviewer: Why do you think you can draw infinite?

Ceylan: Because there are infinite points on the circle. Since they all go to the center point, I can draw infinite congruent line segments

Another focus participant, Ilkan, showed that at the beginning of the interview, he was able to construct the congruent line segment with the help of a compass, but only by determining a single point without determining the possible points. Although he could argue that he could construct an infinite congruent line segment by showing the existence of infinite points on the circle, he had difficulty in putting forward the mathematical justifications. For example, this participant put forward the infinite point on the line segment as a basis as well as the infinite point on the circle. In addition, during the interview, it was observed that while constructing an equilateral triangle, he experienced confusion in performing the steps of other constructions such as constructing an equilateral triangle or finding a midpoint. As a matter of fact, the first interviews were carried out after the teaching experiments for the first three learning goals, and the students were in the teaching process to realize these constructions. However, the confusion about which step to take in which construction strengthens the interpretation that a set of algorithms or operations are applied by rote, which points to static ways of thinking. For this reason, although this participant has actions showing that the thinking process, which has become dynamic in this construction, is progressing in a positive direction, it is thought that it is not at the desired level yet. In the following quote, although Ilkan stated that he could construct infinite congruent line segments, it is seen that he put forward the justification that the line segment also has an infinite number of points while defending this view.

Interviewer: How many other congruent line segments can you draw?

Ilkan: I can draw an infinite number of line segments. All of this is my radius (he has connected another point on the circle with the center point with the help of the straightedge).

Interviewer: Why?

Ilkan: We said before that there are infinite points on the circle. We know that the line segment also has infinite points. Since there are infinitely many, we can draw an infinite number of times.

As a result, in the teaching experiment and clinical interviews, developmental progress was observed for the thinking processes that became dynamic by determining the possible points of the focus participants and making mathematically justified defenses. In the construction of the congruent line segment, it was observed that some students could form a congruent line segment as only parallel to the given line segment, and even with its endpoints one under the other or side by side. This points to a more static thinking process. It was decided to revise the learning trajectory in this direction. Another critical action decided to be added to the learning trajectory is that students should be able to construct the congruent line segments in different ways and directions. Below, the revised part of the learning trajectory demonstrated regarding the construction of the congruent line segment is written in red.

Revised Hypothetical Learning Trajectory

Learning Goal 1: Constructs the congruent line segment with the help of compass and straightedge.

Task 1: Draw a line segment congruent to the given line segment. Build a line segment and transfer it anywhere on the plane.

Conjectured Learning Process

- ♠ Determining two points that are as far away from each other with the help of a compass and constructing the line segment between these points.
  - ♦ Transferring the equal length by opening the compass as much as the line segment and constructing the congruent line segment with the endpoints one under the other, provided that the line segments are parallel.
  - ♦ Transferring the equal length by opening the compass as much as the line segment and constructing the congruent line segments in different ways and directions.

Main Critical Actions Expected in the Learning Process

- S/he becomes aware that the congruent line segment s/he will construct can be in different ways and directions.
- S/he can reveal possible points that are at a specified length from a point by the construction of a circle or an arc.

Construction of Building a Triangle Given Its Three Sides

The course, which is a second teaching experiment, was introduced to the lesson with the task of constructing the desired triangle by presenting three line segments to be accepted as the sides of a triangle. In this construction, it has been observed that there is no difficulty in making sense of the requested process as expected. After this short process, which can be seen as the analysis phase, the students got to work for activities to realize the construction. Firstly, when the individual worksheets of the students who realized that the presented line segments should be transferred, it was observed that they tried to make transfers by combining the line segments end-to-end by trial-and-error. Most of the students tried to make transfers based on a single point instead of determining possible points with the help of a compass. Figure 6, which is presented as an example of this, shows the following explanation *"first I determined a point. Then I measured the first line segment (talking about opening the compass as much as it is) and put it (means I transferred). I also measured the second and third line segments and put them in. I tried until the points (endpoints) coincide"*. Some students, on the other hand, tried to construct the desired triangle by transferring the line segments with the help of a straightedge without using a compass. As an example, the explanation of a student trying to construct a triangle in this direction in intergroup whole class discussions is presented below.

Teacher: How did you do it?

Ilkan: I first transferred this line segment, then the other. Lastly, I transferred this to coincide with that. I did it with the help of a straightedge without a compass.

Other students: But it is forbidden to take a mark on the straightedge.

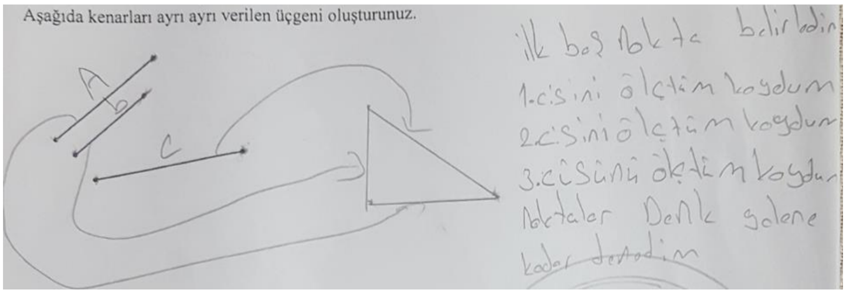

Figure 6. Construction of a triangle with three sides by trial-and-error by making transfers based on a single point using a compass

It was observed that the students themselves had difficulty in finding an acceptable way out, and therefore, the teacher directed them to transfer the line segments by identifying possible points. For this, the following cautionary explanation was directed to them: *"You put a point and try to see if it makes the line segments of that point equal. However, if you put all the points that make it match while transferring the line segments, the point you are looking for will perhaps be revealed"*. Then, it was seen that one student said *"We will find many points"* and the other *"we will do it with a circle"*. However, after transferring the first line segment, which is the side of the triangle, the students had difficulties with where the compass should be placed, and they were not aware of the need to draw a circle. The striking negative situation in this process, which can be seen as an analysis process, is that the students cannot reflect the following strategy, which they put forward while transferring based on a single point, to the transfer process by determining the possible points: *"First, transfer a line segment. Then, based on the endpoints of this line segment, transfer the other line segments in such a way that these two line segments intersect end to end as the third corner"*. It was striking that the students were unable to reason that they should transfer the desired line segment to one endpoint of the first line segment and the other line segment to the other endpoint. It was observed that some students transferred two line segments to the same endpoint (Figure 7).

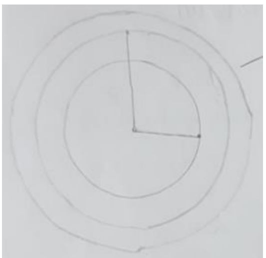

Figure 7. Determining the possible points in the construction of a triangle given three sides, in the direction of transferring the two line segments to only one endpoint of the first line segment.

Thereupon, the teacher posed the following guiding questions and clues on the board: "*Isn't this line segment a side of your triangle?, well then, are the vertices of the triangle clear ?, how many vertices are certain? we need to find it. But the point in this third corner should provide both line segments as sides*". Meanwhile, two of the focus participants, Emre and Ceylan, were able to execute this reasoning quickly and achieve the desired construction. Another noteworthy point is that Emre realized the construction by revealing an infinite number of possible points which are as much as he deems necessary with the help of an arc. This action indicates his dynamic thinking process (Figure 8).

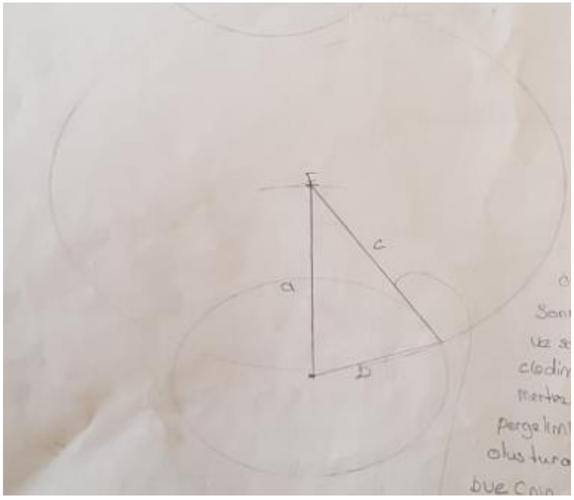

Figure 8. Determining the possible points with the help of an arc when transferring one of the line segments that are accepted as the sides of the triangle

With in-group and intergroup discussions, other students also realized that they had to transfer the two line segments to the endpoints of the first line segment by determining the possible points. Then it was not difficult for them to understand that they would accept the intersection point as the third corner. The process of questioning the feature that makes the intersection point critical is seen as the proving phase. In this process, it was observed that the majority of the class was able to emphasize that the intersection point is the point that provides the desired line segments. It was noteworthy that Ceylan, one of these students, went further and made a justification based on a counter-example situation. The explanation is as follows: "*The third corner of the triangle is the intersection point of the circles. If we put the third corner somewhere else instead of here, c would remain the same, the length of b would be longer (if the third corner was a different point, even if one of the other sides transferred to the endpoints of the first line segment is equal, the other would not be)*". Then, in the process of discussing the steps taken and questioning the existence of different situations that led to the construction, it was noticed that the circle intersection giving the third corner was two. It is accepted by everyone that both points will provide the desired triangle in this process where the proving and the discussion phases are intertwined. Afterward, it is accepted by everyone that the way and direction of the triangle, which is constructed according to the way and direction of the line transferred first, may also differ, but it is the same triangle.

In clinical interviews, Emre, Ceylan, and Salim showed that they could construct the desired triangle by revealing all the possible points as a circle or an arc while transferring all the line segments given (Figure 9).

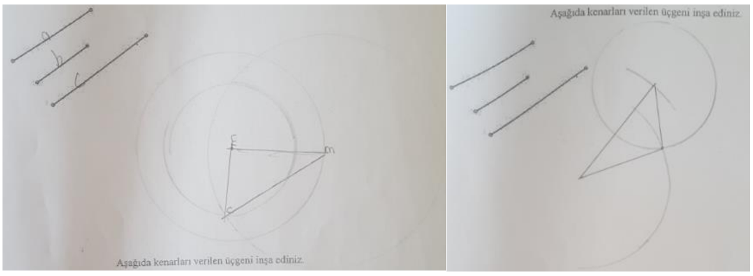

Figure 9. Emre and Salim's determination of the possible points at each step while constructing the triangle given three sides during the interview.

On the other hand, it was observed that Ilkan was able to transfer the first line segment by only revealing a single point with the compass at the beginning, and then put forward unsuccessful actions in transferring the other line segments to the endpoints of the first line segment. Subsequently, he asked permission to transfer the other line segments by the trial-and-error method to intersect the endpoints. But when not allowed in this direction, he probably remembered during the inquiries that he was able to construct the desired triangle by revealing the first line segment by determining a single point and the other line segments by determining the possible points (Figure 10).

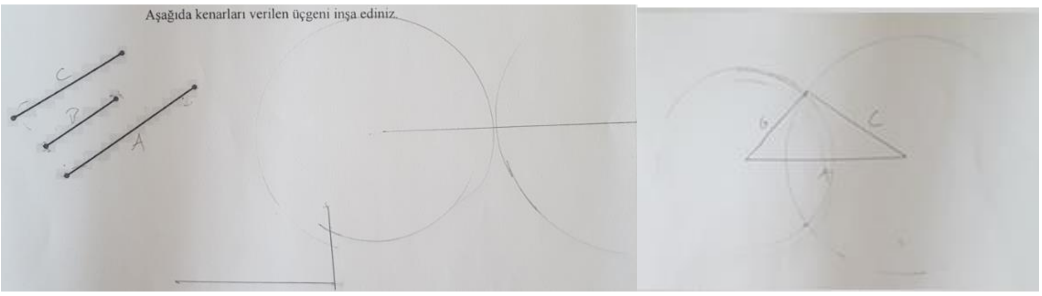

Figure 10. Ilkan's unsuccessful actions in constructing the triangle given three sides (left), and then the construction of the triangle given three sides by determining the single point for the first line segment and the possible points for the others (right).

However, in the question of why Ilkan built a circle, he made the following defense in his question *"not by trial method"* and also why he clarified the intersection points: *"Now it has never been done in the classroom by the trial-and-error. Then Salim drew a circle and made the construction inside it. Emre also found the intersection point by drawing a circle. We learned to do this by seeing him"*. The fact that he presented a non-mathematical justification while defending his actions strengthened the interpretation that he mostly followed algorithmic steps and the thinking process that became dynamic was not yet firmly established. However, other focus participants showed that they were able to make the following argument, which is accepted as a mathematical justification: *"If we took another point on the circle instead of the intersection point, it would provide one side of the triangle but not the other. The feature of this intersection point is that it is the point that supplies both line segments"*. This indicates that they have taken steps by determining possible points and taking into account the ability to create some critical points at the intersection.

As a result, in the construction of a triangle given three sides, two different actions emerged when first transferring any line segment: transferring by placing only one point with the help of a compass and determining the possible points by constructing a circle or arc. In addition, it has been observed that in the transfer of other line segments, an arc is preferred instead of a circle. It was noteworthy that they began to realize that they could benefit from some of the possible points that were deemed sufficient. Another action that can be accepted as one of the important indicators of dynamic thinking is the realization that the constructed triangle may have different ways and directions depending on the position of the first line segment transferred. The relevant parts of the learning trajectory suggested in line with these results, which are deemed to be revised, are presented below in red.

Revised Hypothetical Learning Trajectory

Learning Goal 2: Constructs the triangle when given three sides with the help of a compass and straightedge.

Task 4: The three line segments given are the sides of a triangle. Construct this triangle.

Conjectured Learning Process

- ♠ Transferring any line segment on the plane to the place where the triangle construction will take place.
  - ♦ Transferring by specifying only two points with the compass opening as long as the line segment
  - ♦ Transferring the desired line segment by determining all possible points (circle) or a part (arc) that is considered sufficient for the other endpoint, provided that one endpoint of the line segment is fixed.
- ♠ Transferring the other line segments to the endpoints of the first transferred line segment.
  - ♦ Determining possible points as far as the other line segments separately from the endpoints of the first transferred line segment as a circle (all possible points) or arc (some of the possible points that were deemed sufficient) with the help of a compass.

Task 5: (In relation to Task 4) You found the third corner of the triangle. Are there any other points that could be the third corner? If yes, show it.

Conjectured Learning Process

- ♠ Being aware that the intersection point of the circles or the arcs containing possible points constructed to transfer other line segments to the endpoints of the first transferred line segment is symmetrical based on the first transferred segment.
- ♠ Seeing that the way and direction of the triangle constructed depending on the position of the first line segment transferred may also differ.

Main Critical Actions Expected in the Learning Process

- ♣ S/h realizes that s/he can determine the part of the possible points that he sees sufficient as an arc. In this way, s/he takes an important step towards preventing possible confusion by obstructing the occurrence of intersections that are not critical for the construction.
- ♣ S/he sees that there may be more than one critical intersection point that enables the construction and always considers this possibility for subsequent constructions.

Construction of Finding the Midpoint of a Line Segment

In the learning process that started with the task of finding the midpoint of a given line segment, it was observed that all groups worked towards making sense of the requested construction. It was seen that the students were aware that the line segment is a set of points, and with the advantage of this awareness, they realized that the midpoint is the point in the middle. It was noteworthy that some students embarked on the process of construction without the need to convey to the teacher or make him confirm what they understood from the middle point request. Meanwhile, besides the students whose eye decision determines the midpoint, the students who developed the following idea with the help of compass also attracted attention: *"I have to put the compass in such a place and adjust its opening in such a way that I can draw a circle that is tangent to both extremes. Here, the pointed end of the compass is placed, in other words, the central point is the middle point of this line segment"* (Figure 11). In Figure 11, an image from a student's worksheet is seen in the image on the left, whose eye decision determines the midpoint. In the image on the right, the student's explanation is seen: *"First I drew the line segment. Then I put a dot somewhere on it. I drew a circle (talking about the dot-centered circle). Seeing that it passes through the points (which means from the endpoints of the line segment) I decided that I found the equal distance"*.

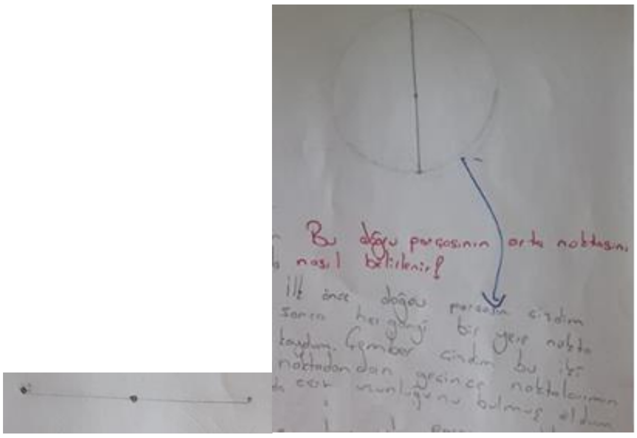

Figure 11. Determination of the midpoint of the line segment by eye decision (left) and finding the midpoint through the eye decision dot-centered circle tangent to the extreme points (right).

The teacher carried both strategies to whole-class discussions. It has been accepted by the whole class that the methods of trial-and-error or eye decision are unacceptable. Then, the teacher emphasized that determining a random point on the line segment and proving that it is the middle point with the help of a compass indicates the methods of trial-and-error or eye decision. The teacher supported them with the following directions to make sense of the critical aspects of the desired construction: *"What is the critical feature of the midpoint?", There are many points on the line segment, but the midpoint has a feature that makes it special, unlike the others. That feature will show you the way out ".* Meanwhile, by saying *"equidistant from both ends"*, it has been observed that the critical feature of the midpoint can be put forward in whole-class discussions. After these processes, which are seen as the analysis phase, firstly Ceylan opened his compass as much as the line segment and drew the circles that accept the endpoints of the line segment as the center and found the midpoint by using the intersection points of the circles. Although this student seems to have taken into account one of the intersection points in this process, which can be clearly seen as the construction phase, it seems that his verbal and written explanations are based on the line segment passing through two intersections (Figure 12). This process, in which the realized construction is questioned, can be seen as a partial transition to the proving phase, but the absolute transition is expected to be in the process of questioning the steps in the construction as to why these intersections are critical.

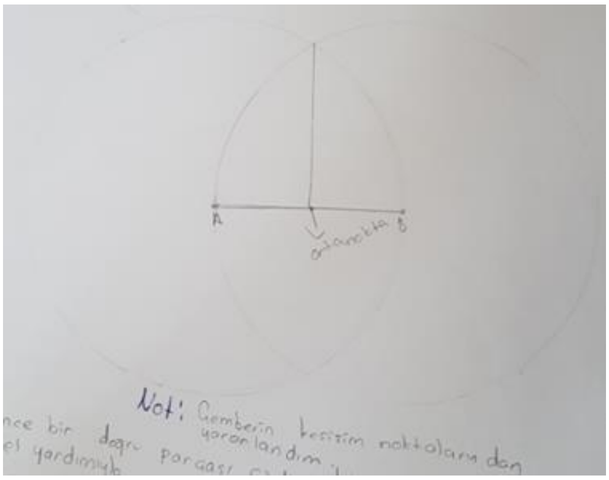

Figure 12. Finding the middle point by opening the compass to the length of the line segment.

After questioning the feature that makes the intersection points critical in the whole-class discussion, it was seen that some of the students thought that they could open the compass only as much as the line segment. Another remarkable point was that many students understood that the midpoint should be equidistant from the ends of the line segment, and in addition, they tried to construct tangential circles that accept the endpoints as the center in order to determine the midpoint (Figure 13).

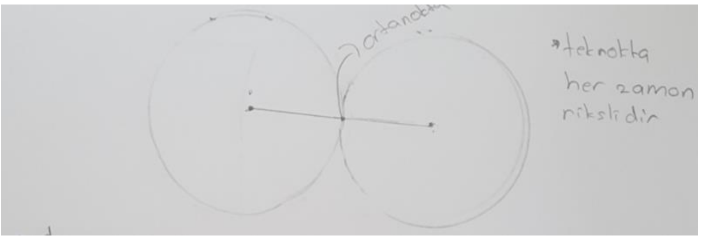

Figure 13. Trying to find the midpoint with the help of tangent circles that accept the endpoints of the line segment as the center.

The teacher emphasized that the accuracy of this construction was considered suspicious because it only revealed a single point, and started the question of whether more points could be found at an equidistance to both ends. In this process, it has come to the fore to realize the construction with the condition that the compass can only open as much as the line segment. However, the process of finding the midpoint in this way does not yet indicate a fully dynamic thinking process, as it imposes restrictions such as being able to open the compass only as much as the line segment, therefore the distance to the endpoints can only be as much as the line segment, and the existence of only two or three points on the perpendicular bisector. For this reason, new thinking and discussion process was initiated by asking the whole class what the common feature of the intersection point of line segments (midpoint) and intersection points of circles (in short, the points on the perpendicular bisector). Thereupon, the simple answers came out at the beginning: *"they provide the intersection points", "they help to find the midpoint"*. In the following time, the discussion was expanded as to whether the opening of the compass could be changed or not, whether it could be opened only as much as the line segment, and whether it could be opened at a different length than the length of the line segment. In this process, which could be considered as a discussion phase, it was noticed that the midpoint could be found with different openings of the compass. In the excerpt of the class discussion presented below, Emre, who is also the focus participant, includes his defense that the compass can have different openings.

Teacher: So, to find the midpoint of a line segment, do you need to open your compass as much as the line segment?

Emre: No. We open the compass at any opening and draw a circle from both ends. We find the intersection points of these two circles and draw a line. Wherever it intersects with the first line segment, that is already the midpoint.

Then, the students were asked to show the intersection points that allow finding the midpoint on the same line segment, and then they were asked whether they could find different intersection points without deleting them at all. In this way, the students saw that the points on the perpendicular bisector have to be at equal distance to the two ends of the line segment separately. With the question of how much they can open the compass at least, they were able to emphasize that they obtain only one intersection point when it is opened half of the line segment and therefore they must open more than half of the line segment. Thus, structuring the construction process that requires finding the midpoint of a line segment with dynamic thinking ways is supported (Figure 14).

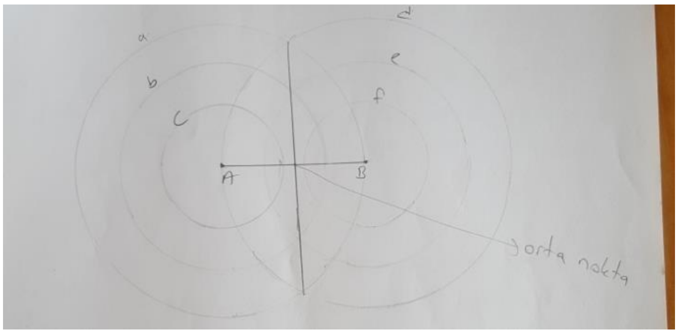

Figure 14. The process of finding the midpoint of the same line segment with a compass opened at different apertures

During the interviews, it was observed that all focus participants were able to find the midpoint of the line segment. However, Ilkan claimed that while finding the midpoint of the line segment, he had to open his compass as much as the line segment. Also, he could not defend the common feature of the intersection point of the circles on the grounds that they are neither in the same direction nor that they are equidistant from the endpoints by being located on the perpendicular bisector. This showed that he has still a static construction image. In the interview excerpt given below, when asked about the common feature of the intersection points, it is seen that Ilkan was insufficient to justify.

Interviewer: So, what do these intersection points (both the intersections of the circles and the midpoint) have in common?

Ilkan: Something straight… being the line segment…

On the other hand, Salim was able to defend the criticality of the intersection points by showing that they are at an equidistant place to the ends of the line segment and that they are located on the line which is the perpendicular bisector. Also, his ability to show that the compass opening can be of different lengths pointed to his dynamic thinking process. He also emphasized that the opening of the compass has not to be shorter than half the length of the line segment, and even explained that it is not acceptable because a single point will appear when it is half the length. However, it was observed that he was unsure whether the compass opening would produce the construction when it was between half the segment and the segment length. In addition, in some cases, the fact that the arcs he constructs reveal only one intersection point instead of two intersections shows the defective aspects of his dynamic construction image. The interview excerpts showing these actions of Salim are presented below.

Interviewer: Well, you opened your compass that much. Could you hold it in a different opening?

Salim: I could. If I open it this far (half the line segment), the midpoint won't be revealed. The circles intersect somewhere here (he shows the midpoint itself on the line segment, the circles will intersect at one point). This is critical intersecting (because of existing a single point). It won't be right. It would be if we opened it larger.

Interviewer: How far can we open the compass to find the midpoint? How far we open it, we can't find the midpoint?

Salim: If we open it small, we cannot find the midpoint. If we open the compass to the length or more of the line segment, we can find it. If we open less than the line segment, we cannot find it (showing that the opening of the compass is unacceptable when it is between half the line segment and the line segment length).

Ceylan, who can interpret and defend the variability of the compass opening, justified the criticality of the intersection points of the circles by being linear only. Although she was clearly asked in the interview excerpt given below, it was seen that she could not show that the intersection points forming the perpendicular bisector are located at an equal distance from the ends of the line segment as a basis that provides criticality in this construction. This shows a deficiency that needs to be repaired in the conceptual basis of the dynamic structure, which is clearly seen that Ceylan is on the way of structuring in the construction of the midpoint.

> *Interviewer: What do these points have in common? Of those… (the midpoint and the intersection point of the circles are asked by pointing finger) So the points at the intersection and this point.*
>
> *Ceylan: These points connect these segments (talking about forming the perpendicular bisector).*
>
> *Interviewer: How do these intersections relate to the endpoints of the line segment?*
>
> *Ceylan:… (silence)*

On the other hand, it was observed that Emre could defend the variability of the compass opening with mathematical justifications. This participant was able to justify the fact that the intersection points are seen as critical in the realization of the construction for different openings of the compass by being at an equal distance from both ends of the line segment and thus being located on the perpendicular bisector (Figure 15). In the meeting, he made the following explanation for the common feature of the intersection points of the circles with the same radius centered on the endpoints of the line segment: *"They are all on the same line. And their distances to these places (to the ends of the line segment) are always the same"*. For this reason, it is thought that Emre takes firm steps towards establishing the dynamic thinking processes he has structured on conceptually solid foundations for the construction of the midpoint.

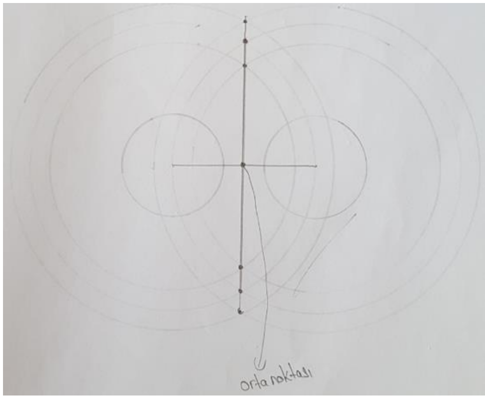

Figure 15. Emre's construction of the midpoint of the line segment by showing the variability of the compass opening.

As a result, it was seen that two different actions that could be put forward for the compass opening in order to determine the points located equidistant from the extreme points, which could not be predicted in the initial learning trajectory, occured. Accordingly, the aspects of the envisaged learning trajectory that need to be revised are given below in red.

Revised Hypothetical Learning Trajectory

Learning Goal 3: Finds the midpoint of a line segment with the help of a compass and straightedge.

Task 6: Find the midpoint of the given line segment.

Conjectured Learning Process

♠ Determining a certain distance and revealing the possible points that are as far away as the specified distance from the endpoints with the help of a compass (building circles with a specified radius, with the endpoints being the center).

- ♦ Determining sets of points that are equidistant from the endpoints, provided that the compass opening is the length of the line segment (the condition that the radius of the endpoint-centered circles is the line segment).
- ♦ Determining sets of points that are equidistant from the endpoints, provided that the compass opening is the half the line segment (the condition that endpoint-centered circles are tangent).

Construction of a Perpendicular Line to a Given Line from a Point not Located on the Line

The lesson in this period of the teaching experiment started with the process of constructing an isosceles triangle. Because the foundation of constructing a perpendicular line to a given another line from a point on the line or not on the line is based on the construction of the height from the vertex to the base in the isosceles triangle. It has been observed that there are five different performances in the construction of an isosceles triangle: *(i) by the trial-and-error method using the only straightedge, (ii) by single-point determination using straightedge and compass, (iii) by constructing the line segments that are accepted as a triangle's sides first anywhere on the plane and then by transferring them through determining possible points, (iv) by constructing a circle and utilizing the congruence of radii, (v) by accepting the line segment as the base, through the intersection of circles of the same radius centered on the endpoints of this base.* It could be comfortably argued in whole-class discussions that the first two ways were unacceptable. The third way, considered acceptable, is valuable in that it allows the reflection of the constructions of transferring the line segment and the building of the triangle given its three sides. The fourth way is an important indicator in the internalization of the circle as the geometric locus of points equidistant from a point on the plane. The fifth way stands out as the most useful way in the construction of the perpendicular line. For this reason, in order to allow students to explore or adopt this way, the following task was presented to them, which would directly support the construction of the perpendicular line: *"Determine the base of an isosceles triangle on a line and construct an isosceles triangle belonging to this base".* Subsequently, this task was expanded by request for determining a point not belonging to the line that would be the vertex of the isosceles triangle. Firstly, three different performances were presented: *(i) creating an isosceles triangle that can determine only one corner of the base on the desired line and therefore unable to construct the base on the line, (ii) using the compass as a measuring tool, putting it on the vertex and leaving one point on the line (constructing an arc or circle with a single point without determining possible points), (iii) thinking of the line as a line segment, creating an isosceles triangle by thinking the arrows at the end of the line as the corners of the base and opening the compass from those ends to the point not belong to the line.* During the whole-class discussion process of these strategies, such guiding questions were asked: *"What is the feature of the vertex?, Is your base on the line?, Where do you intend to construct the two equal sides?, What should be the distance from the vertex to the endpoints of the base?"* With the help of these inquiries, it was seen that the students started to construct the isosceles triangle, whose base is on the line, and whose vertex is the point determined not on the line (Figure 16).

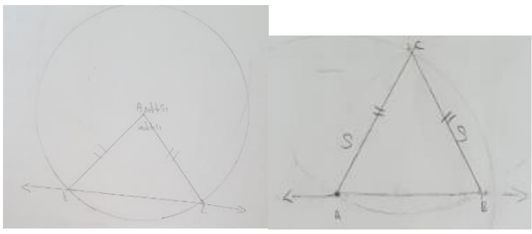

Figure 16. Examples of constructing an isosceles triangle by determining the possible points provided that a line segment is a base and a point not on the line is a vertex

Subsequently, during the task of finding the midpoint of the base of the isosceles triangle, some students noticed that the line passing through the critical points (intersection points of circles or arcs) to find the midpoint also passes through the vertex (Figure 17).

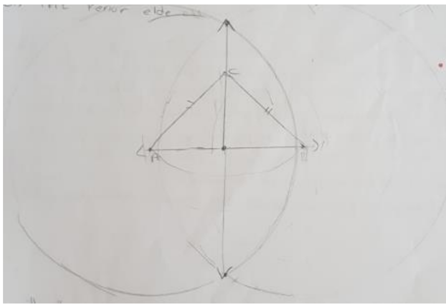

Figure 17. Recognizing that the line that is used to find the midpoint of the base also passes through the vertex, which is the point not belong to the line.

Afterward, the theorem of *"the height from the vertex to the base in an isosceles triangle, that is the perpendicular, divides the base into two equal parts"*, was transferred to the students by the teacher and in this way, they were given the opportunity to reflect on this theorem. In addition, the teacher gave direct support for the interpretation of this theorem as "if a line or line segment passes through the midpoint of the base and the vertex in an isosceles triangle, that line or line segment must be perpendicular". Subsequently, the students were given the task of constructing a perpendicular line that passes through a point not located on that line. Firstly, it was noteworthy that the perpendicular line was built by eye-decision. However, such actions are seen as the analysis phase and therefore considered normal. They were insistently demanded to defend themselves to convince the teacher or themselves whether the line or line segment they had built was perpendicular. This process triggered the transition to perpendicular line construction based on an isosceles triangle. In order to deepen the interaction that started with the discussions within the group, questions such as why an isosceles triangle was constructed, what the point not located on the line and the line segment formed on the line mean in an isosceles triangle, and what are the elements of an isosceles triangle were directed during the whole-class discussions. Besides, the common feature of the vertex, the midpoint of the base, and the intersection points that are used to find the midpoint were questioned in order to internalize the feature of the critical points (especially the point that is also the vertex not located on the line) on the perpendicular bisector. Figure 18 shows that a student who constructed a perpendicular line gave letters on the perpendicular bisector of the base of the isosceles triangle on the individual worksheet and explained that *"all the letters I have given are equidistant to B and C (endpoints of the base)"*.

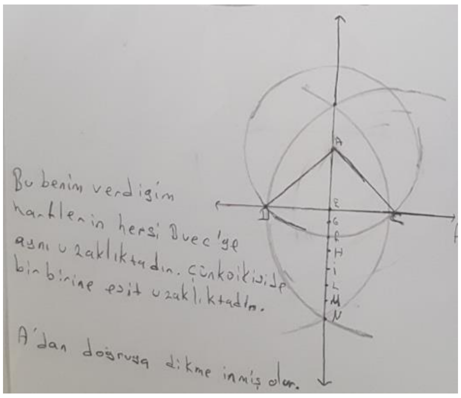

Figure 18. Realizing that the points on the perpendicular bisector including the vertex located at an equidistant to the endpoints of the line segment, which is the base

In order to support the thinking processes that are on the way of dynamization, inquiries were made to interpret the variability of the compass opening and accordingly to see that the base or the equal sides of the isosceles triangle differentiate, and to realize the lower limit of the compass opening enough to cut the line at least two points for the circle or arc that was built first with the vertex center. Also, in these inquiries, which can be thought to have intertwined proving and discussion processes, students were faced with the task of constructing the perpendicular line from points in different places to lines in different ways and directions.

During the interviews, Ilkan was able to construct the perpendicular line only for the horizontal line. However, when asked to build the perpendicular line to a non-horizontal line from a point not located on it, it was observed that he lost control of the steps he took in the construction (Figure 19).

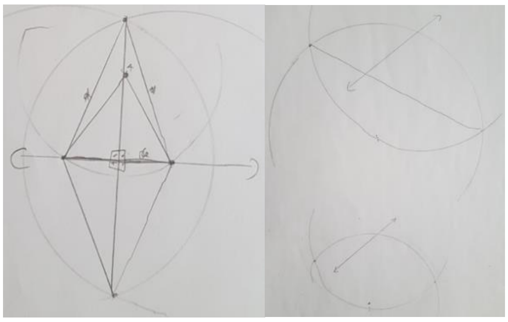

Figure 19. Ilkan's construction of the perpendicular line to the horizontal line (left) but losing control during the construction process for the non-horizontal line (right)

Ilkan showed that he was aware of the existence of an isosceles triangle and finding the midpoint in the construction he realized. These actions are seen as positive steps in terms of the conceptual infrastructure. However, it was observed that this participant could not make defenses by pointing the vertex and base of the isosceles triangle or showing the verticality of the line segment passing through the midpoint of the base and the vertex. Ilkan, who was thought to perceive perpendicularity more visually, gave the impression that he defended the steps in an algorithmic order. Such performances are seen as an important reason for losing control during the construction of the perpendicular line to non-horizontal lines. Below is a sample interview excerpt showing that he could not defend the steps he took in the construction of the perpendicular line by associating it with an isosceles triangle.

Interviewer: So how do you know that this line makes a 90-degree angle here? How can you prove that to me?

Ilkan: This is perpendicular, sir. They intersect here (he shows the point where the perpendicular line and the original line intersect and puts the right angle symbol here).

Interviewer: For example, is there any other geometric shape you've formed here? For example, midpoint... The midpoint of what? For example, does this A (point located not on the line) correspond to something for any geometric shape?

Ilkan: We have built a line (he makes it clear with a pencil that the line means a perpendicular line). It became a line segment if we take it from here (he points the perpendicular line segment between the intersection points of the circles). They need to have the line and line segment to provide the perpendicularity. For example, sir, these intersect here (the intersection point of the perpendicular line and the original line) as plus shape. Since it has a plus shape, here is 90-degree (he puts the 90-degree symbol with a pencil).

However, during the interviews, it was clearly seen that other focus participants could construct the perpendicular line to non-horizontal lines (Figure 20) and also defend their steps through an isosceles triangle (Figure 21). A sample interview excerpt for actions in this direction is presented below.

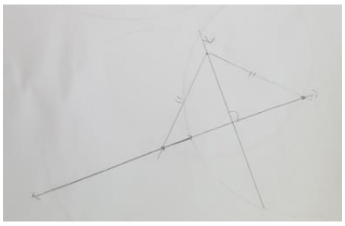

Figure 20. Constructing the perpendicular line to a non-horizontal line through a point not located on the line, provided that the line is extended.

Interviewer: What do these points on the perpendicular line have in common? These points (intersection points are shown).

Salim: They are all equidistant from these points (showing the base vertices of an isosceles triangle).

Interviewer: What is the relationship between being equidistant from those points and being perpendicular?

Salim: For example, if this (the perpendicular line) was a little like this (if it came from a different angle rather than a right angle), then the points would not be equidistant here.

Interviewer: If it wasn't an isosceles triangle, wouldn't it be perpendicular if you did the same thing as you did?

Salim: For example, if the triangle was scalene, it would be like this (showing that the point not located on the line will not be on the perpendicular bisector and therefore that point will not be equidistant from the ends of the line segment). Then the perpendicular line would be like this (trying to show that the perpendicular line will not pass through the midpoint, but will cut the line elsewhere)

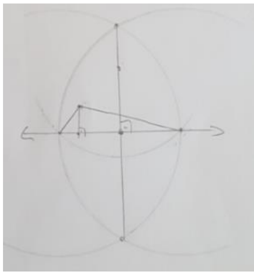

Figure 21. Showing that the perpendicular line will not descend to the midpoint of the base if the constructed triangle is not isosceles.

It was pointed out that the isosceles triangle, which was built as a tool for the perpendicular construction at the beginning of the process, was generally accepted as a tool for proving, without the need for display in later times. For example, in the interview excerpt presented below, it is seen that Ceylan constructs the desired perpendicular line without revealing the isosceles triangle, but when she deems it necessary, she defends by revealing the isosceles triangle (Figure 22).

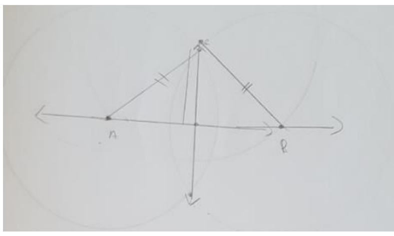

Figure 22. Ceylan's defense of constructing the perpendicular line to the line from a point not located on the line based on an isosceles triangle

Interviewer: So, what is the critical feature of C? C and the intersection points of those circles. What do these three points have in common?

Ceylan: They are all equidistant to points A and B. For example, even if the point would be here (talking about any point on the perpendicular bisector) triangle would always become isosceles (showing an isosceles triangle with a pencil but only passing over it without drawing)

Interviewer: Where is the isosceles triangle? Do you see an isosceles triangle there?

Ceylan: Yes, we get help from an isosceles triangle. Here it is (drew).

As a result, it was seen that during the construction of the isosceles triangle, two different strategies could be put forward, which could not be predicted in the possible learning trajectory. These are constructing an isosceles triangle by reflecting the constructions of building a triangle given three sides and finding the midpoint of a line segment. In this direction, the aspects that need to be revised in the learning trajectory are given below in red.

Revised Hypothetical Learning Trajectory

Learning Goal 4: Constructs the line perpendicular to a given line from a given point not on the line with the help of the compass and straightedge.

Task 8: Draw an isosceles triangle.

Conjectured Learning Process

♠ Constructing an isosceles triangle by constructing line segments that will be sides in another place on the plane and transferring them, in other words by reflecting the constructions of building a triangle given three sides.

♠ Constructing an isosceles triangle provided that the line segment as the base, endpoints of the base as the center for the circles, and the intersection of circles of the same radius as the vertex.

Main Critical Actions Expected in the Learning Process

♣ She realizes that the point not located on the line, which will take the role of the vertex, is located on the perpendicular bisector built during finding the midpoint.

Construction of a Perpendicular Line to a Given Line from a Point Located on the Line

This lesson started with the task of constructing a perpendicular line to a given line from a point on that line. It was observed that students were able to pass the analysis process quickly with the contribution of their experience of constructing a perpendicular line from a point not located on the line. It was pointed out that all groups immediately attempted to realize the construction with compass and straightedge and were aware of the need to construct an isosceles triangle. First of all, it was observed that they had trouble due to the lack of a point not located on the line that would be the vertex. The first step in overcoming this trouble came from Ilkan. Ilkan suggested that the construction of the finding midpoint should reflect for the revealing the vertex. However, it was seen that this student thought of the line as the line segment and the point where the perpendicular line was built as the midpoint of that line segment. However, although he had deficiencies in these aspects, this strategy was the trigger for the step *"I have to construct such an isosceles triangle so that the midpoint of the base is the point where the perpendicular line will pass"*. An excerpt from whole-class discussions illustrating this process is presented below.

Teacher: How can we find the vertex? How should it be?

Salim: There must be such a point that it provides an isosceles triangle.

Ilkan: We put the compass on this endpoint, we draw a circle, then put it on the other end and draw it again (the parts of the line indicated by the arrow symbol, which he calls the endpoint. Therefore, they are not the endpoints of the base. In this way, he considers the line as a line segment and accepts the point where the perpendicular line will be constructed directly as the *midpoint.) But we have to open the compass more than half-length. Then the intersection point of the circles is equidistant from both ends. Here is the vertex.*

Teacher: Well, you said before, the midpoint is equidistant from the two ends of the base. Have you determined those endpoints?

Salim: First, I open the compass this far (think the line as the line segment and accept the parts indicated by the arrow as the end) and put it on the point (the point where the perpendicular line is to be constructed on the line) and draw the circle.

It was entered into the process of noticing the variability of the compass opening with the question "*could the compass be of different opening?*". Meanwhile, after building the base of the isosceles triangle on the line, some students were able to reveal the vertex with the construction of midpoint finding (Figure 23). Then, with the interaction, it was seen that the majority of the students realized that they were able to construct a perpendicular line from a point on the line by taking the similar steps they were familiar with the construction of the perpendicular line from a point not located on the line. In an excerpt from the class discussions presented below, it is seen that the student was able to explain the steps she took in the process of constructing the perpendicular line, noticing the variability of the compass opening.

Teacher: Could you open the compass differently?

Student: I can open it as much as I want. Then I put it on point A (he determined the endpoints of the base and named them A and R) I draw with such opening (he drew an arc, the compass by opening a little more), then I put it at R and draw again. The intersection point becomes the vertex.

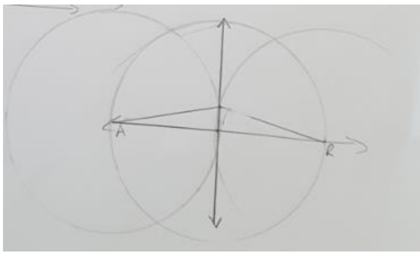

Figure 23. An image of the first examples of the teaching experiment of constructing a perpendicular line to a line from a point located on the line.

In the meantime, the students, who are aware that they reflect the construction of finding the middle point, can easily realize that the line or line segment passing through the vertex and midpoint is the desired perpendicular line. Then, in order to support dynamic thinking processes, the process was expanded with the tasks of constructing the perpendicular line to a vertical line first and then a diagonal line. In this process considered as the discussion phase, the inquiry processes that required them to make defenses based on an isosceles triangle emerged as the proving stage. In Figure 24, it is seen that the participant who constructed a perpendicular line passing through a point on a vertical line could defend his steps through an isosceles triangle with the following explanation: *"First, I drew a line. Then I put my compass at point G and drew a circle of a certain length (radius). I took the intersecting points (between the circle and the line) as the base and opened my compass the distance between the base points. I drew two circles (centered on the endpoints of the base) and thus I have found both the vertex and the midpoint"*

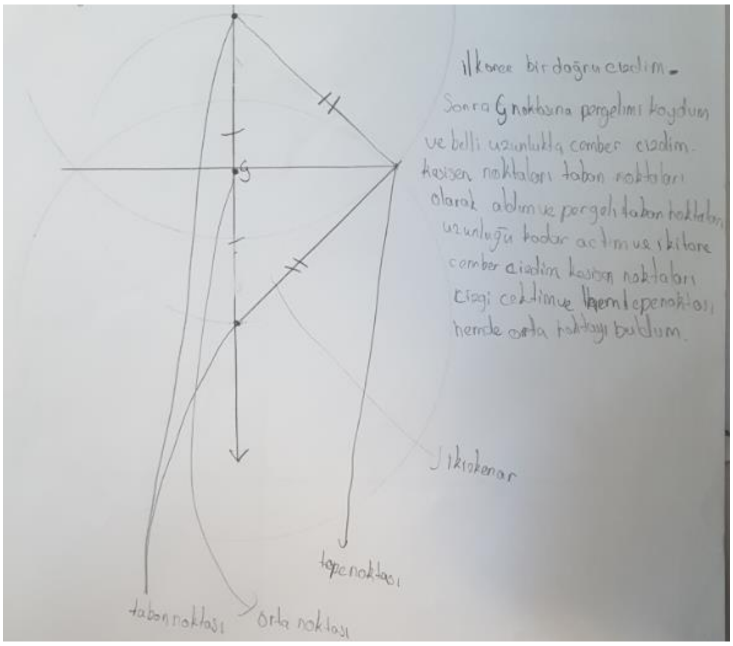

Figure 24. Defending the construction of a perpendicular line passing through a point on a vertical line based on the elements of an isosceles triangle

During the interviews, it was observed that all participants could construct a perpendicular line from a point on both a horizontal and non-horizontal line. However, differences emerged when defending the steps taken in the process of questioning the conceptual foundations. It was seen that Emre and Ceylan could defend with the elements of the isosceles triangle by showing the isosceles triangle construction as a basis. A sample interview excerpt is presented below.

Interviewer: Why did you feel the need to draw such a circle there?

Emre: When I put point A (a point belonging to the line to be built a perpendicular line), I opened my compass as I wanted. Because when I draw with a compass, I find endpoints of the base.

Interviewer: So why did you need to find such a special point (the intersection point of circles centered on the endpoints of the base).

Emre: After I find the base points, when I draw the circles by opening the compass more than half of the line segment, I connect the intersection points and find both the midpoint and the vertex.

It was observed that Salim and Ilkan could draw attention to the existence of an isosceles triangle only when asked directly. It was observed that Salim, who was seen to be unable to directly associate the steps he took with the elements of the isosceles triangle, defended the necessity of the isosceles triangle in the construction of the perpendicular line based on the assumption that *"if it were not isosceles..."*. This participant justified the idea that in a scalene triangle the perpendicular (height) would not go down to the midpoint of the base (Figure 25). His defense in this direction can be seen in the interview excerpt presented below.

Interviewer: How would you explain this perpendicularity with the help of an isosceles triangle?

Salim: Sir, these are equal (he shows the line segments of the vertex to the ends of the base). That's why perpendicular. For example, if it were like this (instead of the vertex, he took a point a little further to the left and formed a scalene triangle. He showed a perpendicular from that point by eye decision) it would still be perpendicular, but there would not be a perpendicular passing through this point.

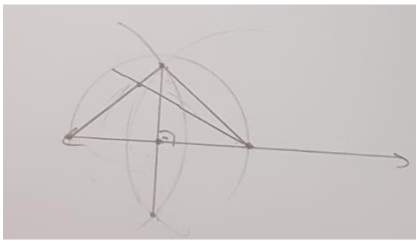

Figure 25. The defense of Salim that the height he built would not intersect with the midpoint of the base if the triangle were not isosceles

Another participant, Ilkan, who was observed to be unable to associate the steps he took with the elements of an isosceles triangle, drew attention to the fact that he made sense of the construction more operationally. It is seen in the interview excerpt presented below that he put forward the justification of *"providing operational steps"* while making his defense.

Interviewer: Why did you first draw a circle on this line?

Ilkan: We had to open our compass a little and draw it to construct the perpendicular line. We had to connect our intersection points with the straightedge. Why did we take these points? Because if we were to get the points from here, the intersection would be here (he says that the point where the perpendicular will be built on the line and the intersection points would not be linear). Maybe it could go through this point again, but it would be a little diagonal and not perpendicular.

Interviewer: So why do you put your compass on these two points (the endpoints of the base on the line)?

Ilkan: If these (arcs) did not intersect, we would not be able to form a perpendicular line by connecting the points.

In line with the results obtained, it was decided that there is no need to revise the envisaged learning trajectory for the construction of a perpendicular line to a given line from a point located on the line.

Discussion, Conclusion and Suggestions

In this study, an iterable and improvable learning trajectory that guides a teaching process that allows the support of the conceptual infrastructure of basic geometric constructions at the sixth-grade level is presented. Although this learning trajectory has been proposed for the sixth-grade level, it is thought that it can also be a reference for higher-level individuals in terms of the results obtained. It provides a practicable road map for revealing dynamic constructions whose conceptual infrastructure is strengthened in learning environments based on the realization of more complex geometric constructions.

Tapan and Arslan (2009) pointed out that teacher candidates mostly used visual elements and presented only experimental justifications in the process of realizing geometric constructions. Similarly, in the process of constructing the first construction that is aimed to realize in this study, it was seen that prohibited actions such as the trial-and-error method, eye decision determination, marking on the straightedge could be preferred without hesitation. But these attempts have entered the process of extinction in later constructions. The functioning of the compass like a measuring instrument due to its opening played a significant role in the students' inclination to realize the formation steps through the compass. Although the act of using the compass as a measuring tool is seen as an experimental justification (e.g. showing that the line segments are at the same compass opening to demonstrate congruence), it was noted that it contributed significantly to the realization of the critical role of the compass in the learning process. It has been observed that some limited actions can be performed even by using a compass. For example, constructing only parallel congruent line segments with endpoints on top of each other, constructing congruent line segments only in the same way and direction, constructing congruent line segments only horizontally. These results support the conclusion of Ulusoy (2014, 2016) that students can perform static thinking ways with the effect of limited sample spaces supported by visual images for the construction of perpendicularity and parallelism. It has been seen that questioning different ways and directions contributes significantly to supporting the dynamization of thinking ways. In this process, the importance of determining all possible points that allow the construction to be realized as a circle or a part of them deemed sufficient, as an arc is remarkable. Instead of determining possible points, students aim to reveal a single desired point many times. This performance leads them to unwanted actions such as trial-and-error. As a matter of fact, it was noteworthy that in the process of construction of a triangle when given three sides, which is the second construction, after the first line segment is transferred, an attempt based on the trial-and-error method is performed to transfer the other two endpoints to join. The criticality of revealing possible points in the realization of this formation and then expanding the reflection process on different ways and directions were clearly seen. However, it has been thought that it would be more appropriate for the students to determine the possible points as a circle first and then to pass on the arc construction themselves over time. Because the ability of students to decide which part of the circumferential points is sufficient improves over time.

It is known that the critical points in the realization of geometric formations can be obtained from the intersection points of the line-the circle, the line-the line, or the circle-the circle (Janičić, 2006; Kellison et al., 2019). In this study, a learning trajectory that supports students to see the intersection points as critical and to defend them with mathematical justifications by realizing the features that make them critical is presented. For example, students who noticed the critical points obtained by the intersection of circles in the construction of a triangle given three sides also realized that they could find the critical point with the intersection of lines in the construction of the midpoint. Besides, realizing that there may be compass openings where circles do not intersect at any point, realizing the construction by interpreting the variability of the compass opening, and putting forward mathematically justified defenses contribute to the support of their dynamic thinking ways. In addition, questioning the common features of the intersection points of circles obtained with different compass openings that make them critical (e.g. all intersection points that are used to find the midpoint form the perpendicular bisector) are considered necessary in order to allow the thinking processes to become dynamic. The dynamic thinking processes of the focus participants from the first three groups could be clearly observed. However, the focus participant from the fourth group was observed to follow only algorithmic steps without mathematical justification, which points to more static thinking processes with the increasing complexity. It is also noteworthy that the focus participant from the fourth group was able to draw a line, line segment, and ray in the open-ended test before the research, but could not provide any explanation for them.

Lim (1997) emphasizes the importance of exposing students to more complex formation problems that can reflect the geometric formations they have already learned, in supporting higher-level thinking skills. Napitupulu (2001) and Kondratieva (2011) draw attention to the necessity of internalizing and reflecting basic geometric constructions in the realization of complex geometric constructions. In the learning trajectory designed in this study, attention has been paid to give an opportunity to reflect the learned constructions during the realization of the other constructions. In this way, it is thought that the internalization of basic geometric constructions can be supported. Since the new construction task presented in the teaching process will require the student to reflect the previous geometric constructions, it is also possible to support the high-level thinking skills pointed out by Lim (1997). For example, in the construction of a perpendicular line passing through a point belonging to the line or a point not belonging to it, it is possible to reflect the constructions of a triangle given three sides, the congruent line segment, and the midpoint. Raising questions to the students to reflect on the existence of these constructions in discussion environments and allowing them to reveal defense processes based on different constructions reflected encourages them to make associations between constructions. In this way, steps towards recognizing the characteristics of a constructed geometric structure and revealing its relations with other structures, whose importance was emphasized by Smart (1998), are supported. During the teaching experiment and interviews, it was observed that the students' ability to defend by putting forward mathematical justifications based on these relationships and characteristics constituted an important basis in evaluating the process of realizing constructions whose conceptual infrastructure was strengthened.

Ulusoy (2014, 2016) and Paksu and Bayram (2019) have shown that middle school students can be more successful in horizontal or vertical lines while constructing the perpendicular line or making judgments and defending the existence of perpendicularity. Similarly, in this study, students' successful actions, especially when constructing a perpendicular line to the horizontal line, drew attention. The positive contributions of the students to make sense of the height construction in the isosceles triangle, which constitutes the conceptual infrastructure of this construction, to be aware of the construction of the finding midpoint they reflect in this process, and to build their justifications on these foundations in order to be able to construct a perpendicular line to non-horizontal or non-vertical lines were clearly seen. In this study, it was seen that Ilkan, who was not able to justify these foundations in the construction of the perpendicular line, could not defend the steps he took and realized the construction more operationally. It is noteworthy that this participant can only construct the perpendicular lines to the horizontal lines, but lose control in other cases. The inability to perform flexible actions such as being able to realize the construction only for certain situations, not defending by diversifying the steps taken, and failing to create alternative ways and not being able to justify them is seen as a deficiency in revealing dynamic ways of thinking. Ulusoy (2014, 2016) showed that the concept images formed by middle school students for perpendicularity include the idea that two vertical lines should average each other, that the lengths of the lines are limited and that the vertical lines should be equal in length, and that only the horizontal and vertical line pairs can form perpendicularity. These results can be interpreted as middle school students' concept images for perpendicularity indicate static thinking processes. In Tapan and Arslan's (2009) study, it was revealed that even teacher candidates could make experimental justifications based on visual elements in the process of realizing geometric constructions. In the teaching process based on the learning trajectory proposed in this study, it was observed that all focus participants except Ilkan were able to reveal alternative ways and make mathematical justifications by going beyond presenting operational constructions supported only by visual images.

Ulusoy (2019) points out that geometric construction activities based on the use of the compass and straightedge are not sufficiently involved in schools. Similarly, Tosun (2019) emphasizes the importance of teachers' lesson planning in a way that allows the use of the compass and straightedge in learning geometry. However, it is seen that the design of the teaching process limited to the analysis and construction phases (Smart, 1998) will be insufficient to support the construction of dynamic constructions whose conceptual infrastructure has been strengthened. As a result of the study, the importance of supporting the cognitive actions such as realizing a construction in different ways and directions, taking into account the changing and unchanging aspects of geometric structures during the construction process, interpreting the variability of the compass opening, revealing all possible points as a circle or part of the points deemed sufficient as an arc, making changes in the route of the steps and interpreting and defending these changes when necessary is seen for realizing dynamic geometric constructions. In order for these cognitive actions to be carried out in learning environments, it is thought that it is important to give students the opportunity to exist in the phases of proving and discussion set forth by Smart (1998). Actions by eye-decision or trial-and-error method during the realization of the constructions are interpreted as the analysis phase. It has been observed that these actions may develop in the direction of determining possible points based on the use of a compass. From this point of view, these actions are valuable in supporting the transition to the construction phase and in preparing the student for the later phases of proving and discussion. The fact that students make a habit of determining possible points instead of realizing constructions based on only one point during the construction phase paves the way for them to make mathematical justifications while defending. This allows them to be present in the evidence and discussion process. At this point, when dynamic geometry software is integrated into learning environments thanks to its functions such as revealing possible points as a circle much faster and more accurately than a compass, making the intersection points of circles with the same radius visible with different color options, tracking variants and invariants of the geometric structure thanks to the dragging function is thought to increase the effectiveness of many students in the process of proving and discussion and directly contribute to the dynamism of thinking processes. Kuzle (2013) and Ulusoy (2019) demonstrated the contribution of the DGS supported learning environment to presenting alternative constructions and verifying the constructions, which directly point to the proving and discussion phases. Also, Ulusoy (2019) concluded that in the DGS supported learning environment, participants realized that they had to make well-founded judgements. When students cannot do this, they can instantly see that their critical characteristics, which are expected to remain invariant when they move the shape thanks to the dragging function, change in the learning environment supported by DGS. In the DGS supported teaching process, it can be much easier for students to be convinced that they are failing to realize the construction. With a compass and straightedge, in a paper and pencil environment, students often use only visual elements (Tapan & Arslan, 2009) or are affected by visual images with prototype shapes (Paksu & Bayram; 2019; Ulusoy, 2014, 2016). For this reason, it can be very difficult for students to be convinced that the desired construction is not achieved in the paper and pencil environment. Lim (1997) and Kunimune et al. (2010) point out that meaningful justifications should be provided for middle school students rather than formal evidence. In this study, it was seen that students could defend their actions in the construction process by providing mathematical justifications when they were provided with a suitable learning environment. The implementation of the designed learning trajectory by arranging it for a DGS supported learning environment means allowing the dynamic movements of the construction and giving students the opportunity to quickly verify the actions and the results achieved. Kuzle (2013) also draws attention to the effectiveness of DGS in the verification process and also showed that they are useful in the teaching process thanks to their functions of providing visualization, specifying precision, and being fast.

In this study, a learning trajectory that focuses on only five of the basic geometric constructions at the sixth-grade level is presented. The need to iterate this learning trajectory and, as a result, continue its revision is obvious. It is also clear that in higher-level classes, the need to design learning trajectories that involve revealing cognitive actions related to the remaining five basic geometric constructions. Besides, it is recommended to conduct studies that investigate the conceptual knowledge of students about the construction and reveal their cognitive actions in learning environments that involve the realization of more complex geometric constructions that require the reflection of basic geometric constructions. Finally, it is thought that the studies focusing on the construction process of basic geometric constructions in a dynamic geometric software environment based on this or a similar learning trajectory will make a significant contribution to the field.

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