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Construction of Inclusion Relations of Quadrilaterals: Analysis of Pre-Service Elementary Mathematics Teachers' Lesson Plans

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Abstract

This research was aimed to find out pre-service elementary mathematics teachers' perceptions on special quadrilaterals and inclusion relations of them and draw their common cognitive paths. In order to reach this aim, this research was designed under qualitative approach and was used document analysis method. The participants of the study were 68 pre-service teachers continuing their 4th year of elementary mathematics teacher education programme at a faculty of education. The data was obtained by the lesson plans which were prepared by the participants. Based on the data, it was determined that the pre-service mathematics teachers had some misconceptions regarding the inclusion relations of quadrilaterals. In addition, it was identified that the "common cognitive path" in terms of quadrilaterals associations was parallelogram / rhombus, square / rectangle and square / rhombus association.

Key words: Inclusion relations of quadrilaterals, common cognitive path, pre-service mathematics teachers

Introduction

Geometric levels of thinking put forth by van Hiele have played an important role in studies related with the teaching of geometric concepts for making sense of the behavior of individuals during the learning process. It has been determined in studies that focus mostly on geometric shapes and inclusion relations that the learning processes of individuals are in accordance with the levels of van Hiele, but that there are more complex cases. In a study, Fujita (2012) has claimed that this complex situation can be overcome by determining the "common cognitive path" drawn by the students regarding the topic. According to Vinner and Hershkowitz (1980), groups understand a topic and its parts in a certain order and this order is called known as the "common cognitive path". According to Vinner and Hershkowitz (1980), learning path and thus error path of students can be the same for some topics. The determination of the common cognitive path can be leading for the teaching of many mathematical concepts and the shaping of programs as well as the arrangement of course plans.

The objective of this study was to determine the perceptions of pre-service elementary mathematics teachers regarding quadrilaterals and inclusion relations as well as putting forth the "common cognitive path" which is the widely used way of thinking.

Studies carried out by a number of researchers on various geometric shapes have put forth a common cognitive structure for the conceptual understanding of individuals. Vinner and Hershkowitz (1980) have stated as a result of their study carried out on triangles that individuals have a common perception that the altitudes of triangles are always inside the triangle and that the common cognitive path for defining the altitudes of triangles falling inside is towards recognizing the altitude drawn outside. Nakahara (1995) concluded in his research conducted on Japanese students that common cognitive path for some quadrilaterals were towards parallelogram, rhombus and trapezoid. Okazaki and Fujita (2007) suggested on their research on Japanese and English students on their perception towards quadrilaterals that students from these two countries had different common

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cognitive paths. They found that the Japanese students comprehended parallelogram/rhombus, square/ rhombus, rectangle/parallelogram, square/rectangle associations whereas English students, differently, followed the common cognitive path of parallelogram/rhombus, rectangle/parallelogram, square/rectangle, square/ rhombus, respectively. Such common cognitive path coming out of the study was related to how individuals have perceptions towards quadrilaterals and how they constructed inclusion relationships.

De Villiers (1994) stated that while they classify geometric shapes and constructing inclusion relations, students prefer partition classification rather than hierarchical classification. According to De Villiers' (1994) description, individuals making partition classification do not take into account inclusion relations of shapes whereas the ones using hierarchical classification can construct inclusion relations and classify shapes accordingly by seeing the associations among the shapes. Partition classification can be a problem for individuals in fully comprehending the inclusion relations of geometric shapes and thus learning geometric shapes. In researches performed about the perception towards quadrilaterals how individuals defined the quadrilaterals, how they construct inclusion relations and how images are constructed have been important. It was seen that such studies were performed for different age groups. It was observed for younger age groups (elementary school level) that individuals have problems in recognizing, naming, defining and mentioning the properties of the shape given. For example, while it was observed that there are individuals insisting that a square in diamond shape is not a square since it is slanted, individuals do not have any difficulty in recognizing and naming the shapes in studies conducted with older age groups but they have problems in describing and constructing inclusion relations among them (Fujita, 2012; Fujita and Jones, 2006; 2007; Okazaki and Fujita, 2007). These problems were mostly encountered in constructing parallelogram associations. In studies, it was found that the individuals could define parallelogram and recognized when they saw it. However, the rate of accepting square, rectangle and rhombus in the same family was not found too high. A similar situation was observed between the association of square and rectangle. Erez and Yerushalmy (2006) found in their study conducted with computer software that since children could not construct associations of quadrilaterals, they did not understand why they use parallelogram key to construct square, rectangle and rhombus. Moreover, in the same study, individuals stated that they had difficulties in relations between square and rectangle by expressing statements such as "I divide the square and obtain a rectangle". Similarly, in their study, Heinze and Ossietzky (2002) observed that more than half of the individuals assumes square as quadrilateral of which all angles are 90° and they saw that there are individuals having a perception that the only quadrilateral having equal sides is square (one third of the participants). According to the findings of some studies it was observed that individuals draw the shape given correctly whereas they cannot define them correctly. For example, it was found that individuals drawing square and rectangle correctly use wrong or incomplete definitions such as "quadrilateral having all sides in equal length" for square and "quadrilateral which has two short and two long sides" for rectangle. It was suggested that there is no mistake in defining the parallelogram as the name has the word parallel and it is associated with the shape (Fujita and Jones, 2007). Relevant studies put forth that individuals do not have difficulty in making parallelogram/rhombus association and the reason is that they resemble each other in terms of appearance (Fujita and Jones, 2007).

It is seen from these studies that the number of studies performed with teachers and pre-service teachers towards quadrilateral perception is limited. It was also observed in studies carried out teachers and pre-service teachers that the findings are similar to those obtained from the studies conducted with students (Okazaki and Fujita, 2007; Türnüklü, Alaylı, Akkaş, 2013). In these studies, it was revealed that the individuals define quadrilaterals in the frame of their own concept images; that they mostly deviate from formal definitions, classify the quadrilaterals according to the images of shapes in their minds and thus make some mistakes. In a study performed with pre-service elementary school mathematics teachers, it was put forth that the number of individuals who could correctly define the quadrilaterals were not high (almost 30%) (Tünüklü, Alaylı, Akkaş, 2013). Furthermore, some findings were also acquired such as seeing rhombus and square in different families, highlighting the feature of sides of rectangle and square being parallel due to typical drawing of parallelogram, and thinking that square and rectangle are in different families. Similar findings were found in different studies performed with mathematics teachers (Tünüklü, Akkaş, Alaylı, 2013).

Theoretical Background

For the determination of the common cognitive path of pre-service elementary school mathematics teachers' common cognitive path regarding quadrilaterals, "concept image" as remarked by Tall and Vinner (1981), "figural concept" proposed by Fischbein (1993) for geometric shapes and "prototype shape" (Hershkowitz, 1990) have mainly come into prominence as the theoretical background.

According to Tall and Vinner (1981) "concept image" and "concept definition" are situations different from each other but have come to the forefront for individuals in forming the concept. Accordingly, concept definition is the grand total of all expressions used to explain this concept. Concept image includes the whole cognitive structure about the concept. These are images of the concept, properties and processes it covers. According to the researchers, individuals' images on the concepts are constructed over the years and show differences.

Naturally, concept and concept image are two elements taking shape together in geometric concepts. According to Fischbein (1993) geometric concepts can be defined by several properties and they also have an image as visual figure. For example, square is effective for individuals in constructing concept image both with its properties (such as equal length of the sides, angles of 90°) and its shape. In this respect, it was observed that individuals are affected from frequently used and common appearances of shapes, i.e. prototypes, in forming concept images (Hershkowitz, 1990). As a result of the researches, it was put forth that individuals shape the definition of geometric concepts with concept image they form in their minds and this may differ from the formal concept definitions (Tall and Vinner, 1981; Monaghan, 2000; Heinze and Ossietzky, 2002; Vighi, 2003). Moreover, concept definitions made in this way play a role in classification of geometric shapes and constructing inclusion relations. It is observed that individuals prefer certain decision making paths for geometric shapes and these paths cause individuals to make some mistakes. These are determined as "prototypical judgment" as seen below:

Type1: Generalization of visual inferences made based on prototype examples to other situations (the ones which do not fit). Example to this type can be; students' thinking the altitude of a triangle inside the triangle, and drawing the altitude inside the triangle and also assume it as altitude in situations which do not fit.

Type2: Using typical properties of prototype shapes in making inferences, decision making and applying them for other types of the concept. For example, claiming that "the shapes other than square are not quadrilaterals, though the sides are equal in length but the angles are not equal" (Hershkowitz, 1990).

In addition to these generalizations another type of generalization was also defined. It is called "analytical judgement". In this type of generalization individuals can make generalizations based on the critical properties that concept includes. In this type decision making individuals can generally make correct decisions. For example, since all the quadrilaterals are closed shapes deciding that a given open shape is not a polygon based on this fact (Hershkowitz, 1990). It was found by some researchers that such types of decision making and generalization explained above cause mistakes in geometric concepts. It was observed that individuals made typical mistakes particularly in determining the inclusion relations of geometric shapes. Fujita and Jones (2007) stated in their study for recognizing and classifying the quadrilaterals that constructing quadrilaterals and their inclusion relations and their developments in this subject might be theoretical in structure when they consider general mistakes and individuals' images they formed towards the quadrilaterals as a basis. Fujita (2012) determined some developmental levels independent of age which are associated with the perception of quadrilaterals and constructing inclusion relations. He classified these levels in the frame of parallelogram family as below (Fujita, 2012: 64):

Hierarchical: Learners can accept squares, rectangles and rhombi are also parallelograms. The opposing direction inclusion relationship of definitions and attributes is understood.

Partial Prototypical: Learners have begun to extend their figural concepts. They accept rhombi are also parallelograms but not squares and rectangles. Their judgment would be likely to be prototypical type 2.

Prototypical: Learners who have their own limited personal figural concepts. Their judgments would be either prototypical type 1 or 2.

0 level: Learners do not have basic knowledge of parallelograms.

It is seen that Fujita's levels formed for parallelogram conform with van Hiele's levels as well. "Visual level" of van Hiele's levels where perceiving the geometric shapes as a whole and naming them according to their appearances overlaps with Fujita's level called "prototype" to some extent. Moreover, "analytic level" and "informal deductive level" of van Hiele's levels show similarities with Fujita's "partial prototype" and "hierarchical level", respectively.

Hershkowitz (1990) claimed that analytical decision making should be supported in order to remove mistakes in geometric concepts. Therefore, he emphasized the need for using the critical properties of the concepts effective in decision making in learning. These critical properties are usually within formal definitions of the concepts (accepted in academic environment). However, both these critical properties, thus formal definitions, typical shape drawings and samples have been effective for individuals in forming their own concept images. Prototype samples of shapes, properties of the family that shapes belong, geometric shapes that have noncritical but effective visual images of more specific examples may direct individuals to form his/her own concept images and thus making mistakes (Fujita, 2012; Fujita and Jones, 2007; Okazaki and Fujita, 2007).

Method

This research aims to put forth the perceptions of pre-service mathematics teachers on special quadrilaterals (square, rectangle, parallelogram and rhombus), their inclusion relations while describing their common cognitive paths. For this purpose, the study was performed by document analysis method based on qualitative research. A document analysis method covers the analysis of written materials which contain information about a phenomenon (Yılmaz and Şimşek, 2008). The documents that form the data of the research are lecture plans prepared by the pre-service elementary mathematics teachers.

The structures reflected in lecture plans that the pre-service mathematics teachers prepared to teach the properties of quadrilaterals comprehensible and explain inclusion relations will draw their general cognitive structures associated with quadrilateral perceptions as well as constructing inclusion relations. In lecture plans collected for this purposes, answers were sought for the questions of what the teaching order and direction of quadrilaterals are, how the relations during teaching are constructed, which critical properties of quadrilaterals become prominent and what the mistakes are.

Study Group

The participants of the study are composed of 68 people studying at mathematics teacher education (4th year) in a public university. The participants were chosen by purposeful sampling method used for qualitative research studies.

Analysis

The analysis of the collected documents (lecture plans) were done within the frame of content analysis. For this purpose, firstly, the whole data were read, arising themes were determined and every text was coded within the framework of these themes. Then, similarities and differences were revealed by comparing both the same codes arising from each individual's plan internally, and the same codes coded in different individuals. Thus, a general structure was intended to be formed. These themes were determined as which properties of quadrilaterals are used, associating the quadrilaterals, association types, whether the relations were constructed correctly and mistakes. In addition, quantitative data analysis was applied by calculating frequencies and percentage of accuracy by counting the associations of quadrilaterals.

Findings

The methods with which pre-service elementary mathematics teachers teach square, rectangle, parallelogram and rhombus by putting forth their inclusion relations differ. Basically, the participants started with one of the quadrilaterals and construct the relations with others over it. It was observed that they construct another quadrilateral which they think it is related to over a quadrilateral they determined in such binary associations or they later relate them by determining totally independently. It was also observed that they use definitions or list the properties while explaining quadrilaterals. The properties emphasized by pre-service teachers were discussed in the findings. Then, the data were evaluated according to how the participants relate which polygon couples to what kind of properties and it was intended to draw a cognitive structure. Some expressions written by the participants were presented to the readers in findings proof. The participants' names were coded by numbers in these expressions and abbreviations such as P.S.T.12 (Pre-Service Teacher 12) were used since they are pre-service teachers.

The properties that can be considered critical which the participants emphasized while defining quadrilaterals or listing their properties have been given in Table 1. The table was constructed by determining what the properties used in determining or describing quadrilaterals were as in expressions such as "I tell that the sides of a square are equal and the angles are 90° (P.S.T. 23)" or "I tell that a parallelogram is a quadrilateral whose sides are parallel (P.S.T. 2)".

Table 1.
Critical Properties of Quadrilaterals

SQUARE	RECTANGLE	PARALLELOGRAM	RHOMBUS
Sides being equal Angles being 90°	Opposite sides being equal Angles being 90°	Opposite sides being parallel Opposite angles being equal	Opposite sides being parallel Opposite angles being equal

As can be seen from Table 1, these critical properties are those that are generally available in text books or proposed to be used to academically to define these quadrilaterals. Among these properties, equal sides have not become prominent much for rhombus though it is included in formal definitions. Because of the name of rhombus in Turkish that means "equal sides", it might be the reason to not mention this property.

According to Hershkowitz (1990) the properties included in the definitions may form a basis for the decision making of individuals. Hershkowitz names these as critical properties. However, noncritical properties might be properties that individuals mostly determine from prototype drawings of shapes and have secondary importance such as, sides of square and rectangle being parallel, diagonal properties of quadrilaterals, sum of internal angles being 360°, sum of supplementary angles being 180°. It was observed that some of the participants used the property of "a quadrilateral that has two short and two long sides" for rectangle. For parallelogram and rhombus, generalization of "angles cannot be right" was made whereas the property of being parallel was not frequently used for square and rectangle. It was determined that noncritical properties are shaped by concept image that the individuals have and this causes mistakes in constructing relations among quadrilaterals. What these mistakes are and how they are related will be presented with associated data.

When the data are evaluated in terms of inclusion relations, the frequency values obtained for the association of the aforementioned quadrilaterals by 68 pre-service teachers have been given Table 2. The table contains frequencies of individuals constructing the quadrilateral on the row using the one given in the leftmost column. When the lesson plans prepared by pre-service teachers are examined, it was observed that some were excluded from the table since they discussed and defined quadrilaterals independently and did not associate them or indicate with which properties they are related to. Therefore, the sum of the frequency values is lower than the number of participants even though the plans of all 68 participants were evaluated.

Table 2
Association Frequency Values of Quadrilaterals

	Parallelogram	Rectangle	Square	Rhombus	*TOTAL*
Parallelogram	----	19	0	14	*33*
Rectangle	13	----	21	0	*33*
Square	1	15	----	15	*31*
Rhombus	1	0	6	----	*7*
TOTAL	*15*	*34*	*27*	*29*	----

It can be seen from the frequency values in Table 2 that most of the participants (13 people) define and construct a parallelogram by associating it with a rectangle. It is not too common to construct it using square (1 person) and rhombus (1 person). Angle property became prominent in parallelograms defined as based on rectangle or constructed by a property of rectangles. It was observed that some of the participants were explaining angle property of parallelogram by comparing it to rectangle made mistakes as shown in the expression below.

"I want students to see that parallelograms are rectangles tilted at an angle (P.S.T. 32)."

It is observed that the image, called prototype, associated with the typical drawing of parallelogram typically used in explaining the shape is effective for this participant. However, the individual made a mistake due to this prototype image effect. This expression leads one to the generalization that the angles of a parallelogram would never be 90°. This indicates a Type 2 generalization determined by Hershkowitz (1990). Similar cases are also seen in the participants' expressions shown below.

"After I explain a rectangle I draw a parallelogram and compare them. So, I show that the angles of a parallelogram are not right but its opposite angles are equal (P.S.T. 45)".

"I tell the students that in contrast with a rectangle, the internal angles of a parallelogram, are not equal even though it has equal opposite sides (P.S.T. 49)".

It was observed that they made mistakes in constructing a parallelogram by using a rectangle. However, a different association was observed in two of the participants. They pointed out the angle property and they brought a different perspective to the relation between a parallelogram and a rectangle by the expression "its angles are not constant". Such a relation removes the mistake mentioned above.

It was observed that one participant associated parallelogram using rectangle based on the equality of sides. This pre-service teacher caused a misconception to occur by associating two quadrilaterals with the expression, "I emphasize that sides of a parallelogram are not equal though all sides of rhombus are (P.S.T. 61)". As can be seen from this expression, the prototype image of a rhombus and parallelogram caused an erroneous association. According to this cognitive approach all sides of a parallelogram cannot be equal and thus rhombus cannot be included in the parallelogram family.

It was observed that rectangles are constructed mostly by associating them with parallelograms (19 people) and squares (15 people) as acquired from the frequency values presented in Table 2. The participants who constructed rectangles using parallelograms generally highlighted the angle property. The examples are the following expressions: "when the angles of a parallelogram are made 90° it becomes rectangle (P.S.T. 12)", "The form at which the angles of parallelogram is made 90° is a rectangle (P.S.T. 28)". In this approach, rectangle became a special type of parallelogram in terms of angles. However, mistakes appeared in some expressions when highlighting the angle of 90° as a *different* property such as, "the difference of rectangle is that the angle is 90° (P.S.T. 14)" or "I compare the angles of rectangle and parallelogram and they see the difference (P.S.T. 20)".

It was determined that the participants construct rectangles using square based on the side property. It was expressed that the sides of a rectangle are not equal in length or it is rectangle when opposite sides are identical but have different length. There are some expressions below in which mistakes can be clearly seen.

"After I state the properties of a square I tell that when we increase the length of two sides equally it becomes a rectangle (P.S.T. 62)".

"Different than a square, all sides of a rectangle are not equal (P.S.T. 6)".

Constructing rectangles based on squares and building relations between them cause no mathematical problem. However, this will cause problems in constructing inclusion relations as in the expressions above. That is because square and rectangle cannot be considered within the same family in such a situation. Particularly, since the second participant tried to explain the situation between two quadrilaterals by using the term *different* caused a mistake. In this case, analogy of "square is a special rectangle" cannot be formed in the mind of an individual.

As can be seen from the data in Table 2 based on the expressions of participants, squares were mostly constructed based on rectangle (21 people). It is observed that relation is constructed via the side property. No mistake was mostly encountered in an association in this direction.

"I explain squares based on the fact that the sides of a rectangle become equal (P.S.T. 60)".

"First a rectangle is drawn. It is observed that its sides are perpendicular and that its opposite sides are equal to each other, then the special rectangle with all sides equal, i.e. square is given (P.S.T. 3)".

"Square is a rectangle with all sides equal (P.S.T. 5)".

"When we make all sides of a rectangle equal we obtain a square (P.S.T. 12)".

Another quadrilateral from which square is constructed is rhombus. Although construction of a square based on rhombus is rare (12 people), association was mostly based on the angle property.

"I make them notice that when the angle of a rhombus is 90° it becomes a square (P.S.T. 44)".

"I first explain a rhombus then shift to square and tell that square is a special type of rhombus (P.S.T. 48)

As the values in Table 2 are analyzed it is observed that parallelogram and square was used in construction of rhombus. Side relation was became prominent in the expressions about rhombus constructed based on parallelogram.

"Rhombus is a special form of parallelogram the sides of which are equal (P.S.T. 22)".

"Rhombus is the form of a parallelogram with equal sides (P.S.T. 5)".

Moreover, there were participants who associate or construct rhombus with square. They usually built a relation by pointing out the angles. Examples to these expressions are "I make them draw a square by changing the angles but without ruining the sides and then I tell that it is a rhombus (P.S.T. 11)" and "the difference of a rhombus from a square is that its angles do not have to be 90° (P.S.T. 9)". It was observed that some of the participants who highlighted the angle relations between square and rhombus made mistakes. Two participants' expressions given below are examples to this.

"I ask what kind of shape it would be if the angles of a square are not right (P.S.T. 4)".

"When we change the angles of a square we obtain a rhombus (P.S.T. 24)".

As can be seen from the expressions of these pre-service teachers, the prototype image of a rhombus causes problems in relating it to a square. As in the expressions here a perception can come up pointing out that a rhombus can never have an angle of 90°.

As the frequencies given in Table 2 are taken into account without considering the relation direction it is seen that most of the participants (36 people) associated rectangle with square. Then comes association of parallelogram with rectangle (32 people), square with rhombus (21 people), and parallelogram with rhombus (15 people). When it is analyzed that whether these associations are correct, percentage values given in Table 3 are obtained. Square and parallelogram relation is not given in the table since it was not encountered (there were only one participant associating them, however this person did not write it clearly).

Table 3.
Percentages of Associating Quadrilaterals without Mistake

Parallelogram/Rectangle	Parallelogram/Rhombus	Square/Rhombus	Square/Rectangle
75%	93%	57%	86%

As the whole lesson plans prepared by the participants are taken into account no individual teaching square, rectangle, parallelogram and rhombus with all inclusion relations was found. When making binary associations, relations of quadrilateral couples with other quadrilaterals were neglected. As this is evaluated in terms of classification relations defined by De Villiers (1994) it can be said that an explanatory style were encountered in which a partial hierarchical classification is done. It was observed that a small number of participants followed an explanatory style by only mentioning the properties of the quadrilaterals without constructing any relations.

Discussion and Conclusion

It can be said that pre-service mathematics teachers use properties specified as critical in constructing square, rectangle, parallelogram and rhombus as well as the associations among them. Moreover, noncritical properties were used by the participants in associating some quadrilaterals and this caused them to make interpretations as if both quadrilaterals do not possess this property though they actually possess it. Neglecting that these square and rectangle are also parallelograms and not remembering that there are parallel sides are examples for this. However, it is obvious that in using critical and sometimes noncritical properties they construct inclusion relations which will cause some mistakes due to the effect of conceptual images. For example, while the right angles of a square is a critical property, the fact that this is not a critical property for a rhombus caused a perspective to develop in individuals such as "rhombus is a square without right angles".

It is observed that mistakes presented in such findings are due to generalization mistakes called Type1 and Type2 defined by Hershkowitz (1990) as stated in the background section of this article. They can be summarized as:

For Square-Rectangle association: Different than squares in terms of side lengths of rectangles can never be equal.
For Parallelogram-Rectangle association: Different than rectangles in terms of angles of a parallelogram can never be 90°.
For Parallelogram-Rhombus association: Different than a rhombus in terms of sides of a parallelogram can never be equal.
For Square-Rhombus association: Different than a square in terms of angles of a rhombus can never be 90°.

Such erroneous generalizations resemble the findings of other studies (Berkün, 2011; De Villiers, 1994; Fujita, 2012; Trünüklü, Akkaş, Alaylı, 2013). As all these are considered the participants corresponds to Fujita's "partial prototype" and to some extent "prototype" level in constructing inclusion relations. These levels can assume square, rectangle and rhombus in parallelogram family and indicate that participants can make generalization mistakes due to concept image and prototype shape image mentioned above. If this finding is associated with van Hiele's levels it can be said that the participants are in "visual" and to a large extent in "analytical" level.

The fact that the relation between square and parallelogram was never constructed resembles the findings of another study conducted with mathematics teachers (Türnüklü, Akkaş, Alaylı, 2013). Furthermore, similar results were obtained in the study mentioned in which a situation was encountered refusing that rhombus and parallelogram, and square and rhombus have a special status.

As all these mistakes and association styles are taken into account it is seen that there is a problem in constructing negative inclusion relations pointed out by researchers (Erez and Yerushalmy, 2006; Heinze and Ossietzky, 2002; Okazaki and Fujita, 2007). In other words, for example parallelogram is a family covering rectangles, each rectangle has parallelogram property. However, a parallelogram may not have rectangle properties. Individuals that cannot comprehend this situation, i.e. the ones not constructing negative (opposite) inclusion relations correctly may reach a generalization that may cause mistakes as in the following expression: "quadrilaterals having angles which are not right are parallelograms". Hence, such expressions give rise to the findings of this study for each quadrilateral relation. These types of mistakes overlap with "analytical level" of van Hiele's levels.

Of course not all generalizations were wrong. It is considered that some associations provide clues to how some of the mistakes can be prevented. For example, it was more meaningful to relate squares as having a special status based on rectangles. This perspective leads to an inference that assumes squares are in rectangle family and that square is a special form of rectangle, and this was put forth correctly by the individuals. Secondly, the expressions in the form of "in contrast, its angles are not constant" which does not create the perception for some participants that the angles of rhombus and parallelogram can never be 90° provide some educational clues.

As a conclusion, when all such associations and to what extent they have been inferred without mistake are taken into account, it is possible to draw the cognitive path as shown in Figure 1.

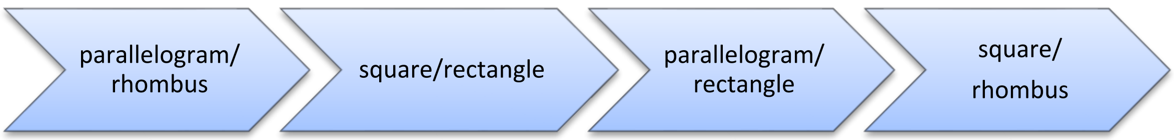

Figure 1: *"Common Cognitive Path" in Constructing Inclusion Relations in Special Quadrilaterals*

It can be basically said in this path that individuals mostly construct parallelogram/rhombus and then square/rectangle, parallelogram/rectangle and finally square/rhombus associations. It is seen that the common cognitive path presented in this study resemble the ones given by Okazaki and Fujita (2007) for Japanese students and English pre-service teachers to some extent. Parallelogram/rhombus association was placed on the top both in this study as well as in Okazaki and Fujita's (2007) study. As stated by a number of researchers in this study these two quadrilaterals are also the ones constructed most successfully since their prototype shapes resemble a lot (Fujita, 2012; Okazaki and Fujita, 2007) and they do not cause any problem in associating the critical properties. The other quadrilaterals in the order differed from Okazaki and Fujita's study. Parallelogram/rectangle association which was on the second place in Okazaki and Fujita's study is on the third place in the current study according to the data. In this study, by mostly focusing angles, the participants' definition for the difference between parallelogram and rectangle in the form that the angles of parallelogram being not 90° made the participants be mistaken. It can be said that there is a cognitive obstacle here. Fujita (2012), Okazaki and Fujita (2007) also indicated similar mistake in their studies. Parallelism relation constructed in parallelogram/rhombus relation can be used as an analogy for parallelogram/rectangle relation. The differences arose in terms of common cognitive path in this study might have been resulted from cultural differences.

Determination of common cognitive path in constructing inclusion relations in special quadrilaterals was useful in revealing how and to what extent the participants correctly constructed the relations. By this way it can be possible to prevent the probable conceptual mistakes by constructing relations among quadrilaterals and form correct concept images in education of these subjects. Moreover, an opportunity can be given to pre-service teachers to propose the reasons of mistaken concept images which can be encountered in teaching these subjects. The data obtained will be important in determination of "common cognitive path" by performing researches, without being limited with this study, with pre-service mathematics teachers as well as different age groups and constructing polygons and their inclusion relations, and thus solving problems. In this context, this study was limited in terms of the way to obtain data and working with a limited number of people. However, on educational basis, it is considered that the findings of the study provide some clues in terms of both directing the educations of pre-service teachers as well as the teaching of mathematics.

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